Aperiodicity for Rational Series

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SAMSA
Warsaw, 2026-06-16

Aperiodic Semigroups

Definition.
$\forall x \in S, \exists n \geq 1, x^n = x^{n+1}$

APERIODIC SEMIGROUPS

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$$(s,t) \cdot (u,v) = (s,v) \to x^2 = x$$

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Non-aperiodic semigroups

$\mathbb{Z}/2\mathbb{Z} \rightarrow x + 1 \neq x$

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Definition.
$\forall x \in S, \exists n \geq 1, x^n = x^{n+1}$

Equivalent definition (finite).
$\exists n \geq 1, \forall x \in S, x^n = x^{n+1}$

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Aperiodic $\neq$ not periodic

Non-aperiodic semigroups $(\mathbb{Z}, +)$ **Aperiodic semigroups**

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Aperiodicity in Automata Theory

Star-free

Counter-free

FO-definable

Aperiodic

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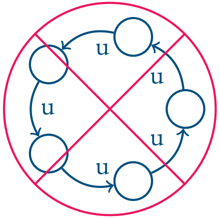

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Aperiodicity in Automata Theory

$$L := a \mid LL \mid \Sigma^* \setminus L \mid L + L$$

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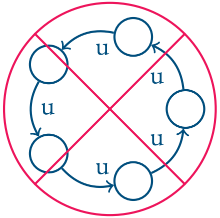

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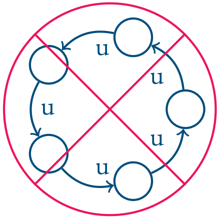

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Σ^* / L is an aperiodic semigroup

APERIODICITY IN AUTOMATA THEORY

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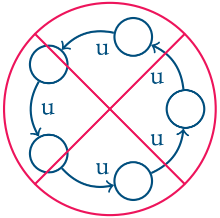

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$\varphi \in \mathsf{FO}[P_a, \leq]$

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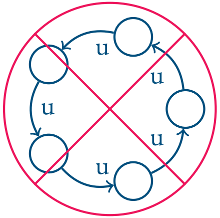

Star-free ⟷ Counter-free

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Σ^* / L is an aperiodic semigroup

What About Rational Series ?

$\Sigma^* \to \mathbb{Q}$. **Recall** $\{a\}^* \simeq \mathbb{N}$

$n \mapsto 2^n$	indicator function of a periodic regular language ?
$n \mapsto (-1)^n$	closed under sums ?
$n \mapsto 2 \times (-1)^n$	closed under products ?
$n \mapsto n^2$	closed under linear combinations ?
$n \mapsto (-2)^n$	closed under Kleene star ?

Decide if Aperiodic or Not

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Past Attempts

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JOURNAL OF ALGEBRA **66**, 448–483 (1980)

Séries formelles et algèbres syntactiques

Christophe Reutenauer

LITP, Institut de Programmation, 4 pl. Jussieu, 75005 Paris, France

Communicated by P. M. Cohn

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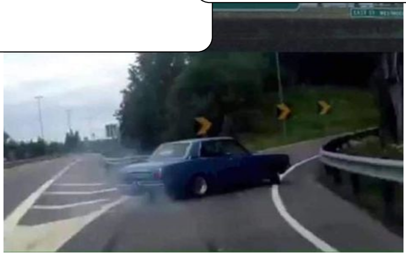
Proposition III.4.4. *Soit* $S \in k\langle\langle X\rangle\rangle$, *rationnelle. Les conditions suivantes sont équivalentes*:

(i) *Tout quotient simple de* $\mathfrak{M}_S$ *est isomorphe à* k.

(ii) *Toute représentation matricielle de* $\mathfrak{M}_S$ *est triangulable.*

(iii) S *appartient à la sous-algèbre de* $k\langle\langle X\rangle\rangle$ *engendrée par les lettres et les séries géométriques.*



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$n \mapsto (-1)^n$

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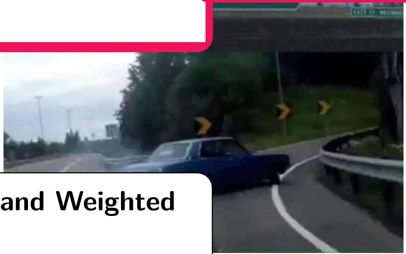

Aperiodic Weighted Automata and Weighted First-Order Logic

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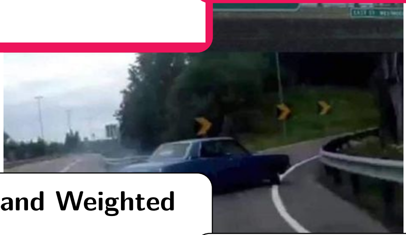

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$$\varphi ::= \top \mid P_a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x \varphi \quad \text{(FO)}$$

$$\Psi ::= r \mid \varphi\,?\,\Psi : \Psi \quad \text{(step-wFO)}$$

$$\Phi ::= \mathbf{0} \mid \textstyle\prod_x \Psi \mid \varphi\,?\,\Phi : \Phi \mid \Phi + \Phi \mid \textstyle\sum_x \Phi \quad \text{(wFO)}$$

with $a \in \Sigma$, $r \in \mathsf{R}$ and x, y first-order variables.

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$\varphi ::= \top \mid P_a(x) \mid x < y \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x \varphi$ (FO)

$\Psi ::= r \mid \varphi ? \ldots$ (step-wFO)

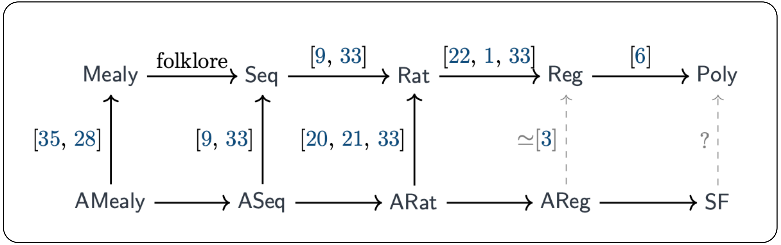
$\Phi ::= \mathbf{0} \mid \prod_x \ldots$ (wFO)

$\prod_x (-1) \equiv n \mapsto (-1)^n$

with $a \in \Sigma$, $r \in \mathsf{R}$ and x, y first-order variables.

Aperiodic Functions

$\Sigma^* \to \Gamma^*$



Aperiodic Functions

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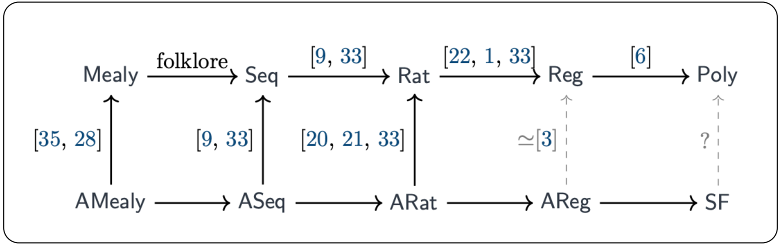
Mealy = (Flip-Flop ∘ Group)*

Rat ≃ Hom ∘ Mealy^R ∘ Mealy

Poly ≃ Square + Reg



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APERIODIC FUNCTIONS

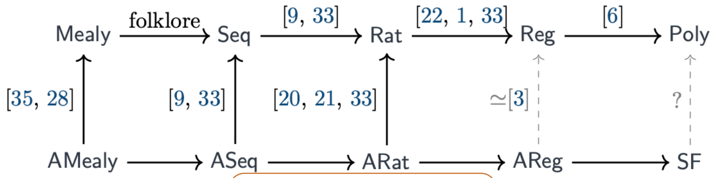
$\Sigma^* \to \Gamma^*$

Mealy $=$ (Flip-Flop $\circ$ Group)*

Rat $\simeq$ Hom $\circ$ Mealy^R $\circ$ Mealy

Poly $\simeq$ Square $+$ Reg

Aperiodicity is orthogonal to computing power



$X \cap AY = A\min(X, Y)$

Seq $\simeq$ Hom $\circ$ Mealy

Reg $\simeq$ Map^R $+$ Map $+$ Rat

$\mathbb{Z}$-polyregular functions…

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INFORMATION AND CONTROL 5, 91–107 (1962)

Finite Counting Automata

M. P. Schützenberger*

$\mathbb{Z}$-polyregular functions...

2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science

Quantitative Monadic Second-Order Logic

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ℤ-polyregular functions…

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Quantitative Monadic Second-Order Logic

Science

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SELECT COUNT(*) FROM WHERE.
$f(w) = |\{\alpha \mid w, \alpha \models \varphi\}|$

M. P. Schützenberger*

$\mathbb{Z}$-polyregular functions...

$\mathbb{Z}$-polyregular functions

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Formalism	Characterization of $\mathbb{Z}$Poly	Characterization of $\mathbb{Z}$SF
Counting formulas	Counting valuations in MSO (Definition II.5)	Counting valuations in FO (Definition V.1)
Polyregular functions	sum $\circ$ polyregular (Proposition II.13)	sum $\circ$ star-free polyregular (Proposition V.17)
$\mathbb{Z}$-rational expressions	Closure of rational languages under Cauchy products, sums, and $\mathbb{Z}$-products (Theorem II.20)	Closure of star-free languages under Cauchy products, sums, and $\mathbb{Z}$-products (Theorem V.4)
$\mathbb{Z}$-rational series that are/have	Ultimately N-polynomial (Theorem II.31)	Ultimately 1-polynomial (Theorem V.13)
	Polynomial growth (Theorem II.31)	n/a
	Eigenvalues in $\{0\} \cup \mathbb{U}$ (Theorem II.31)	Eigenvalues in $\{0, 1\}$ (Theorem V.18)
Residual transducer	Residual transducer (Corollary IV.19)	Counter-free residual transducer (Theorem V.13)

TABLE I: Summary of the characterizations of $\mathbb{Z}$Poly and $\mathbb{Z}$SF expressed in different formalisms.

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$f(uv^n w) = poly(n)$ for large enough n

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M. P. Schützenberger*

WHAT SHOULD BE TRUE

$$\Sigma^* \to \mathbb{Z}$$

Pumping Lemma.
$f(uv^n w) = polyexp(n, \lambda_1, \ldots, \lambda_k)$
for large enough n
where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^+$

Spectrum.
Eigenvalues of *a* matrix semigroup computing f are in $\mathbb{R}^+$

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Decidability.
For unary input ? For commutative input ?
For $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$?

Syntactic characterisations.
Transductions ? Weighted logics ? Star-free expressions ?

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