

# Optimal sequential flows and a short-ends factorisation theorem

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## Sequential Flow Problem [Colcombet, Fijalkow, Ohlmann '20]

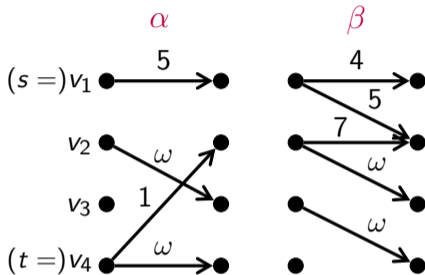
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A *tile* is a function  $V \times V \rightarrow \mathbb{N} \cup \{\omega\}$   
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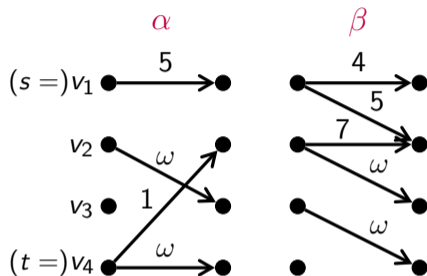
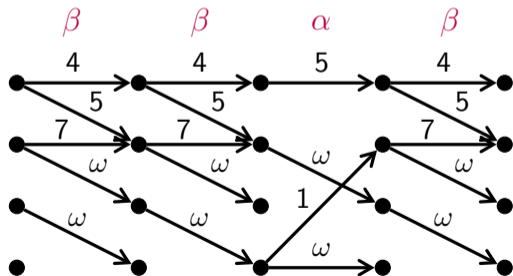
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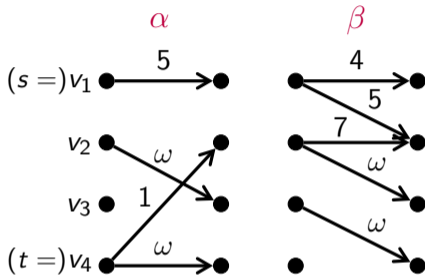
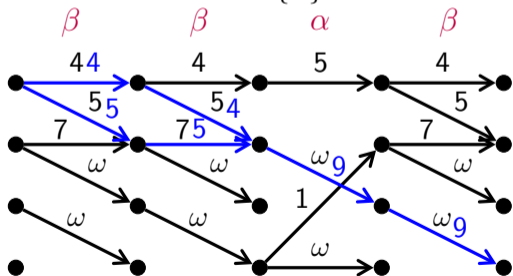


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MaxFlow : **Tiles**\*  $\rightarrow \mathbb{N} \cup \{\omega\}$

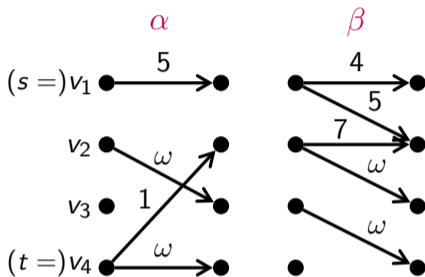
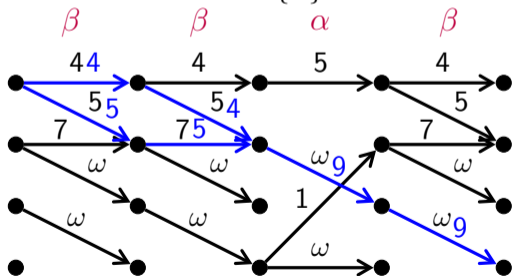


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## Problem

**Input:** A set of tiles  $\mathbf{Tiles}$

**Output:** Is  $\{\text{MaxFlow}(w) \mid w \in \mathbf{Tiles}^*\}$  unbounded?

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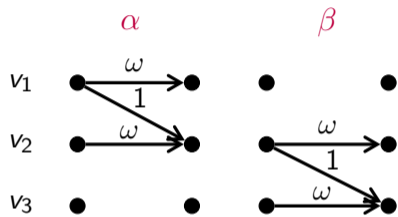
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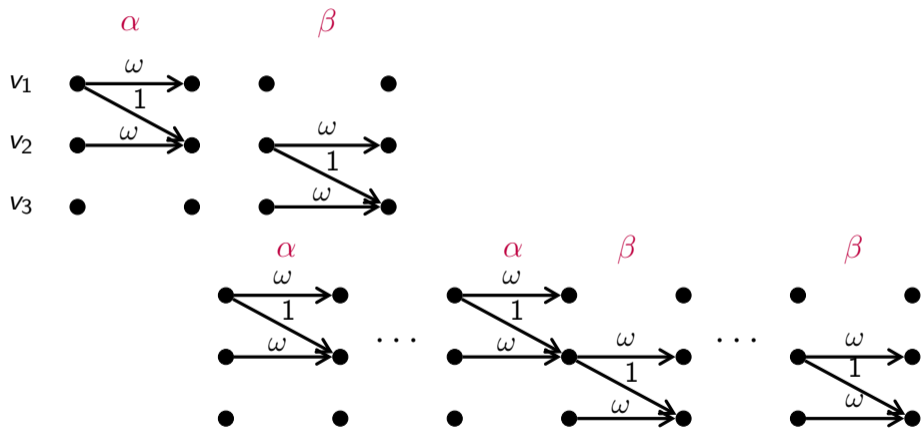
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**Output:** *Compute*  $\sup\{\text{MaxFlow}(w) \mid w \in \mathbf{Tiles}^*\}$ .

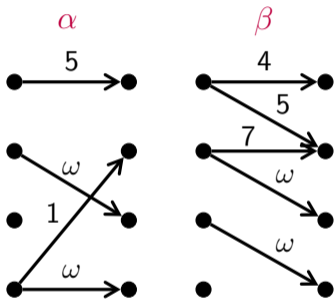
# Example



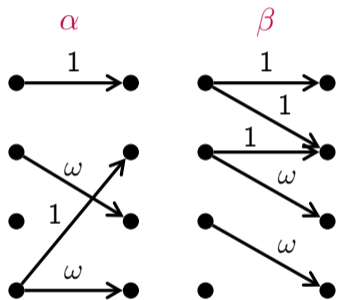
# Example



# Abstraction

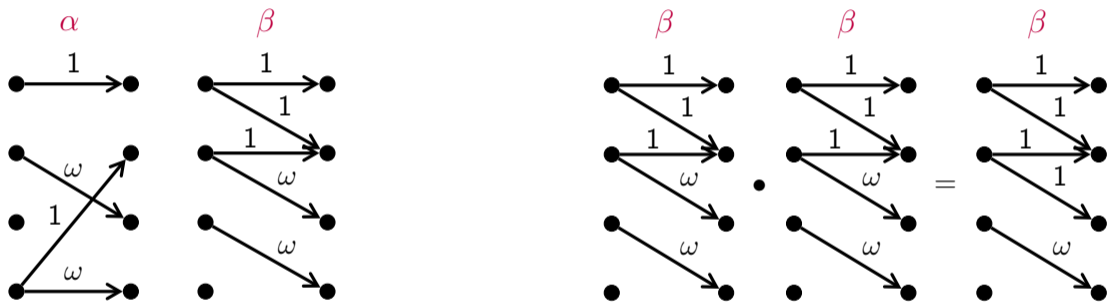


# Abstraction



Morphism  $\varphi : \mathbf{Tiles}^* \rightarrow \{0, 1, \omega\}^{V \times V}$  with max-min product.

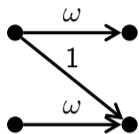
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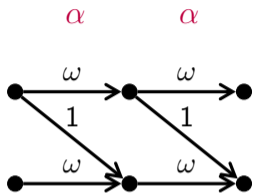
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## Iteration (Inspired by work of Simon, Hashigushi, Leung)

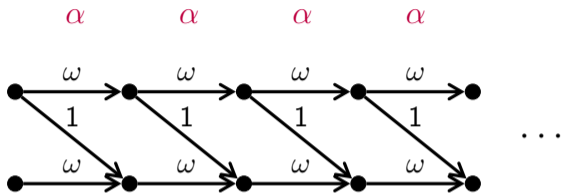
$\alpha$



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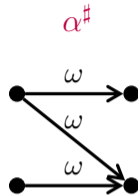
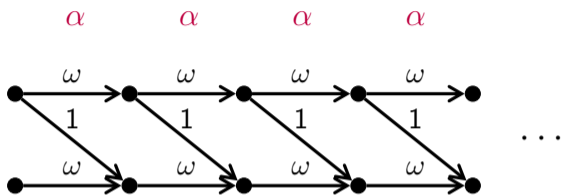


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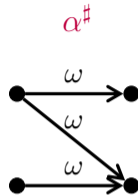
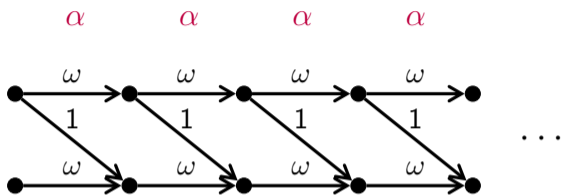


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If  $\alpha^\sharp(s, t) = \omega$  then we have unbounded flows between  $s$  and  $t$ .

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**What about the other direction?**

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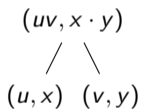
*Factorisation tree*: Node labels in  $\Sigma^* \times \mathbf{M}$ .

**3 types of nodes:**

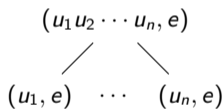
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$(a, \varphi(a))$

**Product nodes**



**Idempotent nodes**



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**Leaves**

$(a, \varphi(a))$

**Product nodes**

$(uv, x \cdot y)$   
/ \  
 $(u, x)$   $(v, y)$

**Idempotent nodes**

$(u_1 u_2 \cdots u_n, e)$   
/ \  
 $(u_1, e)$   $\cdots$   $(u_n, e)$

### Theorem

Every word has a factorisation tree of height  $\leq 3|\mathbf{M}|$

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Proof by induction on the height of factorization trees.

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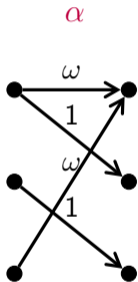
Proof by induction on the height of factorization trees.

**But how large can a bounded flow be in terms of nb of vertices  $n$ ?**

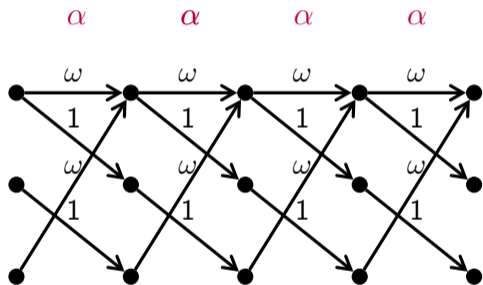
Using Simon's theorem  $\rightarrow 2^{3|\mathcal{F}_\#|} \leq 2^{3n^2+1}$

Can we do better?

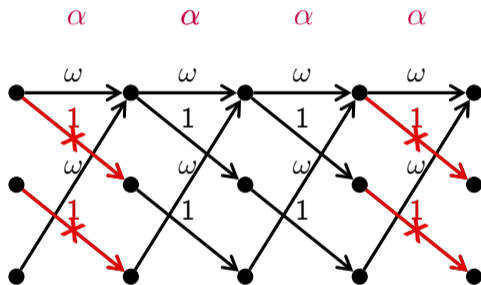
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## Iteration dichotomy



For all idempotent  $\alpha$  and  $s, t \in Q$ ,

- ▶ either  $\alpha^\sharp(s, t) = \omega$
- ▶ or there is a bounded *cut* between  $s$  and  $t$  in the first and last  $\alpha$ .

## Short-ends factorisation theorem (based on work by [Simon '90], [Colcombet '11])

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*Short-ends factorisation tree:*

**Leaves**

$(a, \varphi(a))$

**Product nodes**

$$\begin{array}{c} (uv, x \cdot y) \\ / \quad \backslash \\ (u, x) \quad (v, y) \end{array}$$

**Idempotent nodes**

$$\begin{array}{c} (u_1 u_2 \cdots u_n, e) \\ / \quad \backslash \\ (u_1, e) \quad (u_n, e) \end{array}$$

$\varphi(u_i) = e$  for all  $i$ .

## Ramsey bounds (based on work by [Jecker '21])

$R_{\mathbf{M}}(k)$  = minimal  $n$  such that every word  $w$  of length  $n$  has a factor  $u_1 \dots u_k$  with  $\varphi(u_1) = \dots = \varphi(u_k) = e$  an idempotent of  $\mathbf{M}$ .

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### Theorem

*For all  $w \in \Sigma^*$ , there is a factorization tree of height  $\text{poly}(\log(\mathbf{M}), \log(R_{\mathbf{M}}(3)))$ .*

### Corollary

*In a transition monoid  $\{\top, \perp\}^{n \times n}$ , every word has a factorization tree of height  $\text{poly}(n)$ .*

## Exponential bound

- ▶ Every word has a factorisation of polynomial height in  $n$ .
- ▶ If there is a bounded cut between  $s$  and  $t$  in a sequence of tiles, then there is one that lives within the factorisation.

### Theorem

*If the flow between  $s$  and  $t$  is bounded then it is at most exponential in  $n$ .*

## Applications

**Short-ends Factorisation Theorem**

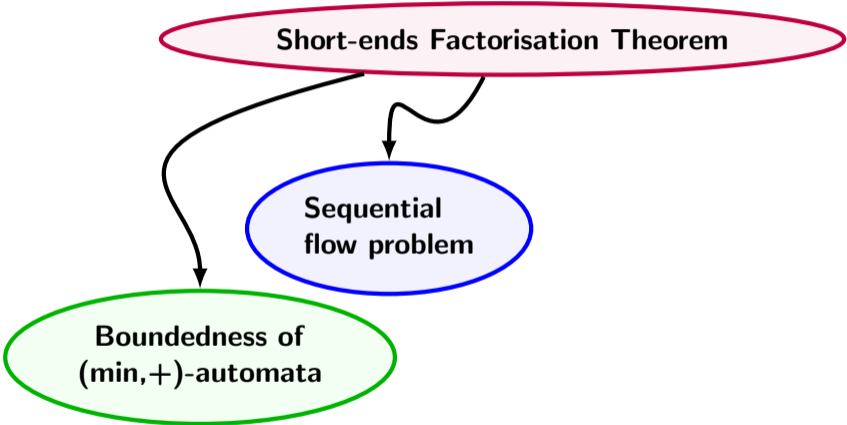
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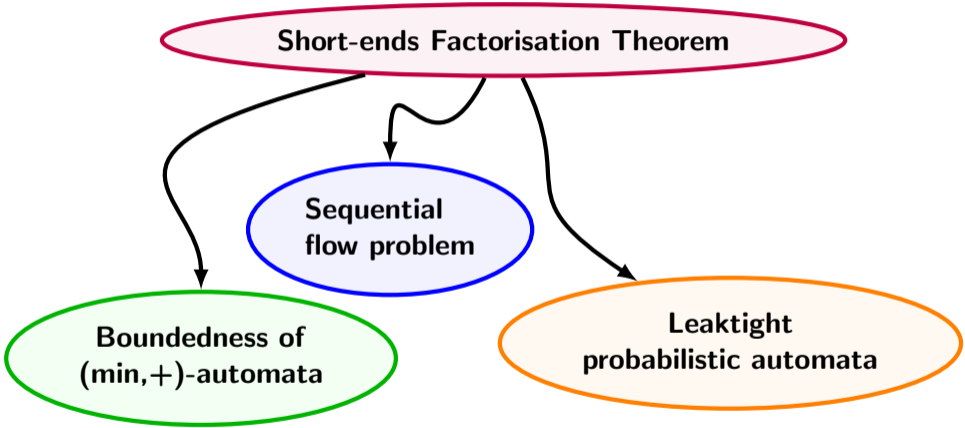
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graph TD; A([Short-ends Factorisation Theorem]) --> B([Boundedness of (min, +)-automata]);
```

**Boundedness of  
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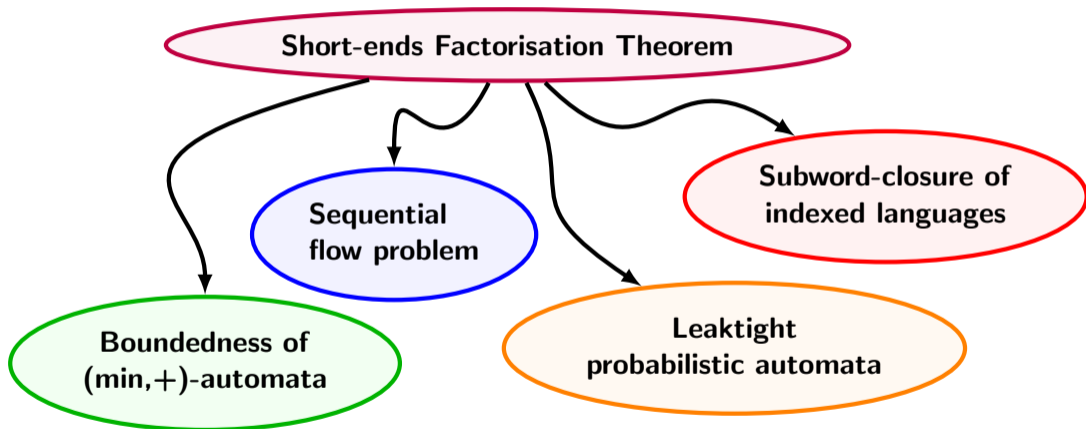
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## Theorem

Let  $\varphi : \Sigma^* \rightarrow \mathbf{M}$  a morphism.

Every  $w \in \Sigma^*$  has a short-ends factorization tree of height  $\text{poly}(\log(\mathbf{M}), R_{\mathbf{M}}(3))$ .

## Corollary

Let  $\mathbb{M}_n = \{\top, \perp\}^{n \times n}$  and  $\varphi : \Sigma^* \rightarrow \mathbf{M}$  a morphism.

Every  $w \in \Sigma^*$  has a short-ends factorization tree of height  $\text{poly}(n)$ .

*Thanks!*