

The Status of the Ambiguity Hierarchy of Weighted Automata over Fields

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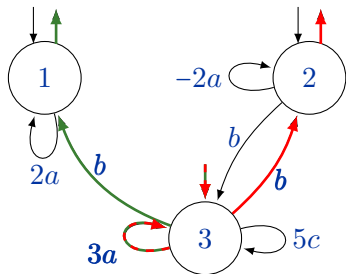
Overview

1. Weighted Automata (WFA) & Ambiguity Classes
2. Determinizability and Unambigualizability
3. Invertible Case: Beyond Unambigualizability

The Model: Weighted Automata

Example

X alphabet, (S, \oplus, \odot) semiring, e.g. $S = (\mathbb{Q}, +, \cdot)$ (or $(\mathbb{Z}, \min, +)$).



$$f(a^2b) = 3 \cdot 3 \cdot 1 + 3 \cdot 3 \cdot 1 = 18,$$

$$f = 2 + 2b + 8a^2 + 6ab + 2b^2 + 10cb + \dots + 18a^2b + \dots - 60b^3ababcb + \dots \in \mathbb{Q}\langle\langle a, b, c \rangle\rangle$$

$(\mathbb{Z}, \min, +)$ case:

$$f(a^2b) = \min\{3 + 3 + 1, 3 + 3 + 1\} = 7.$$

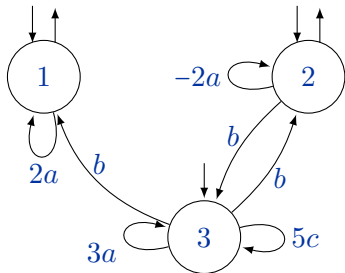
Computational model

WFA computes (=recognizes) a **rational** $f: X^* \rightarrow S$:

- ▶ Given $w \in X^*$, find all successful runs for w .
- ▶ On each run, take the product (\odot) of all the weights, then sum (\oplus) over all runs.

$S = \mathbb{B}$: unweighted automata (NFA).

Weighted Automata



A weighted automaton (WFA):

- ▶ finite set of states Q
- ▶ initial weights $I: Q \rightarrow S$
- ▶ terminal weights $T: Q \rightarrow S$
- ▶ transitions $E: Q \times X \times Q \rightarrow S$.

Transitions have **labels** (letters) and **weights** (elements of S).

Represent WFA using **linear representation**

(u, μ, v) of dimension $d = |Q|$:

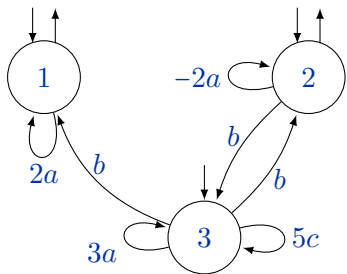
- ▶ $u \in K^{1 \times d}$ (initial weights)
- ▶ $v \in K^{d \times 1}$ (terminal weights),
- ▶ $\mu(x)$ transition matrix of the letter x : row i contains the outgoing weights from state i .

Extends to $\mu: X^* \rightarrow K^{d \times d}$ a semigroup homomorphism, i.e., $\mu(x_1 \cdots x_k) = \mu(x_1) \cdots \mu(x_k)$.

Output for $w = x_1 \cdots x_l$ is

$$f(w) = u\mu(w)v = u\mu(x_1) \cdots \mu(x_l)v.$$

The example as linear representation



$$u = (1 \quad 1 \quad 1), \quad v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A = \mu(a) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \mu(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$C = \mu(c) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Evaluation:

$$f(a^2b) = uA^2Bv = 18$$

$$f(b^3ababcb) = uB^3ABABCv = -60.$$

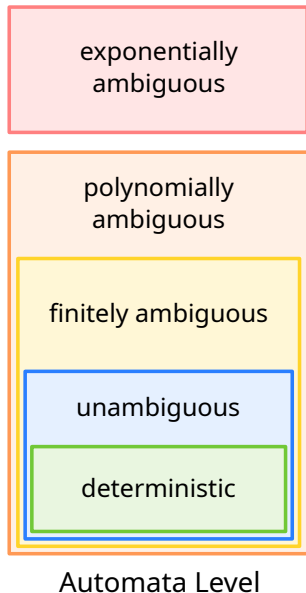
⇒ Finitely generated matrix semigroup $\mu(X^*) \subseteq K^{d \times d}$ controls much of the behavior.

Ambiguity

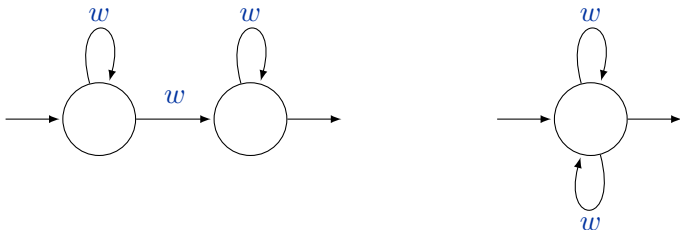
Ambiguity Hierarchy

- ▶ \mathcal{A} is **deterministic** (=sequential) if $I(q) \neq 0$ for at most one $q \in Q$, and for every $p \in Q, x \in X$, there exists at most one q with $E(p, x, q) \neq 0$;
- ▶ \mathcal{A} is **unambiguous** if every $w \in X^*$ has at most one successful run;
- ▶ \mathcal{A} is **finitely ambiguous** if there exists $M \geq 0$ such that every $w \in X^*$ has at most M successful runs (M -ambiguous);
- ▶ \mathcal{A} is **polynomially ambiguous** if there exists $k \geq 0$ such that each $w \in X^*$ has at most $O(|w|^k)$ successful runs.

If \mathcal{A} is not polynomially ambiguous, it is **exponentially ambiguous**: there exists words w, u, u' such that $uw^k u'$ has at least 2^k successful runs.



Deciding Ambiguity



- ▶ For a given automaton the ambiguity class does not depend on the semiring.
- ▶ Efficiently decidable ([Weber–Seidl 1991](#)).

Changing the automaton?

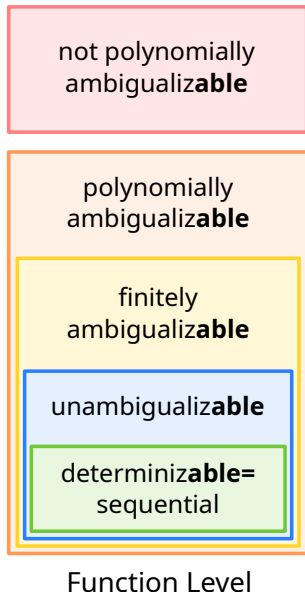
Main question

For a rational $f: X^* \rightarrow S$ (given by a WFA), is there a deterministic, unambiguous, finitely ambiguous, polynomially ambiguous WFA recognizing f ?

- ▶ $S = \mathbb{B}$: every NFA is equivalent to a DFA (subset construction).
- ▶ What if S a field? ($S = \mathbb{Q}$)
- ▶ What if S a tropical semiring? ($S = (\mathbb{Z}, \min, +)$)

Remark

In general, classes are all distinct.



Fields vs. Tropical Semirings



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VS.



(CC-BY 4.0, by Vyacheslav Argenberg)

The tropical case

- ▶ Several partial results (e.g., decidability for 1-valued, unary, finitely ambiguous) tropical automata (Choffrut 1977/78, Gaubert 1994, Mohri 1997, Klimann–Lombardy–Mairesse–Prieur 2004).
- ▶ Survey (Lombardy–Sakarovitch 2006) on determinizability (=sequentiality) results up to that point.
- ▶ Determinizability & unambigualizability are decidable for **polynomially ambiguous** min-plus automata over $\mathbb{Z} \cup \{\infty\}$ (Kirsten–Lombardy 2009).
- ▶ Determinizability & unambigualizability are decidable! (Almagor–Arbel–Sheinvald 2026). Primitive recursive complexity bound.

Fields



Goals

1. **Semantic characterization** of ambiguity hierarchy: what property of rational $f: X^* \rightarrow K$ characterizes the existence of an unambiguous, finitely ambiguous, polynomially ambiguous WFA recognizing f ?
2. **Structural/Algebraic Characterizations** of the Ambiguity Hierarchy.
3. **Decidability** of the ambiguity hierarchy: given a WFA \mathcal{A} , is it decidable whether there exists an equivalent WFA \mathcal{A}' such that \mathcal{A}' is deterministic, unambiguous, etc.?

Algebraically: Minimal WFA

Theorem

Every WFA has minimal linear representations (of minimal dimensions), which is computable in PTIME (for $K = \mathbb{Q}$).

If (u, μ, v) is a linear representation of f and (u_0, μ_0, v_0) is a minimal linear representation of f , then there exists a (rectangular) matrix T such that

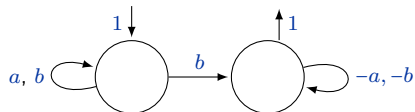
$$u_0 = uT, \quad \mu(x)T = T\mu_0(x), \quad \text{and} \quad Tv = v_0.$$

- ▶ Idea: restrict to the reachability set $\text{span}\{u\mu(w) : w \in X^*\}$, and factor out the non-observable part $\cap_{w \in X^*} \ker(\mu(w)v)$.
- ▶ Any two minimal linear representations are conjugate by an invertible T (so minimal linear representations are unique up to isomorphism).
- ▶ After base change:

$$u = \begin{pmatrix} * & u_0 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} * & 0 & 0 \\ * & \mu_0 & 0 \\ * & * & * \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ v_0 \\ * \end{pmatrix}.$$

- ▶ Minimal linear representation $\not\Rightarrow$ least ambiguity.

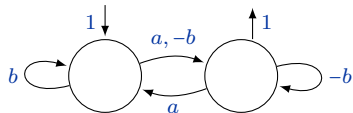
Determinizable or Not?



$$u = (1 \ 0),$$

$$\mu(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mu(b) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$



$$u = (1 \ 0),$$

$$\mu(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(b) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

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Determinizability

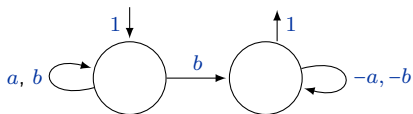
Theorem

Let $f: X^* \rightarrow K$ be rational with minimal linear representation (u, μ, v) .

1. (**Algebraic**) f is determinizable if and only if $u\mu(X^*)$ can be covered by a finite set of lines (through the origin). (? Raney 1958).
2. (**Semantic***) f is determinizable if and only if it is unambigulizable and of bounded variation (Bell- S. 2021).
3. Determinizability is decidable (essentially Hrushovski-Ouaknine-Pouly-Worrell s2019, Zariski closure of a finitely generated matrix semigroup).

*: Semantic characterization of unambigulizability: later.

Determinizable or Not?

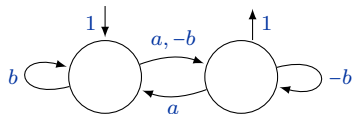


$$u = (1 \ 0),$$

$$\mu(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mu(b) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$u\mu(X^*) = \{(1, \pm n) : n \in \mathbb{N}_0\}$ cannot be covered by finitely many lines.



$$u = (1 \ 0),$$

$$\mu(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(b) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$u\mu(X^*) = \{(\pm 1, 0), (0, \pm 1), \pm(1, -1)\}$ is even finite

Proof Sketch

Proposition

If f is determinizable, then $u\mu(X^*)$ can be covered by a finite set of lines.

Proof.

Let $(\widehat{u}, \widehat{\mu}, \widehat{v})$ be a (trim) deterministic linear representation of f .

\widehat{u} has at most one nonzero entry $\Rightarrow \widehat{u} = \lambda e_i$ for a standard basis vector e_i .

Each matrix $\widehat{\mu}(x)$ has at most one nonzero entry in each row.

$\Rightarrow e_j \widehat{\mu}(x)$ is either 0 or $\lambda' e_l$ for some l .

$\Rightarrow \widehat{u}\widehat{\mu}(X^*)$ is covered by lines Ke_1, \dots, Ke_d .

Minimization gives a linear map that sends $\widehat{u}\widehat{\mu}(X^*)$ to $u\mu(X^*)$, so $u\mu(X^*)$ is covered by finitely many lines! □

Proof Sketch

Proposition

If $u\mu(X^*)$ can be covered by finitely many lines, then f is determinizable.

Proof: each line becomes a state.

Example

$u = (1 \ 0)$, $\mu(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mu(b) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $u\mu(X^*) = \{(\pm 1, 0), (0, \pm 1), \pm(1, -1)\}$

Let $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, -1)$.

New 3-state automaton:

$$u' = (1, 0, 0), \quad A' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v' = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Unambigualizability

Unambiguizability, semantically

- ▶ Let $\Gamma \leq K^\times$ be the finitely generated group generated by all weights of \mathcal{A} , and $\Gamma_0 := \Gamma \cup \{0\}$.
- ▶ Every successful run produces an output in Γ .

Pólya property

If \mathcal{A} is unambiguous, then all outputs are in Γ_0 with $\Gamma \leq K^\times$ finitely generated (**Pólya property**).

Theorem (Bell-S. 2021)

Let $f: X^* \rightarrow K$ be rational. TFAE:

1. f is unambiguizable;
2. f is Pólya: there exists a finitely generated $\Gamma \leq K^\times$ such that $f(X^*) \subseteq \Gamma_0$.
3. a **linear hull automaton** of f is unambiguous.

Was conjectured by Reutenauer 1979; one-letter case (LRS) known by Pólya 1921, Benzaghrou 1970, Bézivin 1987.

Computability of the Linear Hull

Corollary

Computing the linear hull gives decidability of unambigualizability / determinizability!

Sufficient: (linear) Zariski closure of a finitely generated matrix semigroup (polynomial invariants):

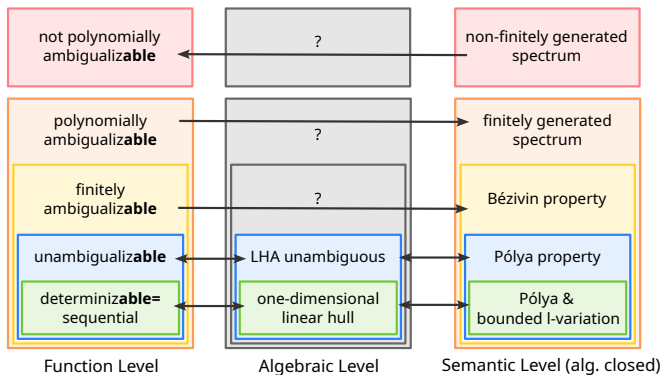
- ▶ [Derksen–Jeandel–Koiran 1995](#) (group)
- ▶ [Hrushovski–Ouaknine–Pouly–Worrell 2018](#) (semigroup)
- ▶ [Nosan–Pouly–Schmitz–Shirmohammadi–Worrell 2021](#) (group variant)
- ▶ [Bell–Smertnig 2023](#) (linear; double-exponential output size)
- ▶ [Benalioua–Lhote–Reynier 2024](#) (linear; **2-EXPTIME** bound)

Theorem

1. Determinizability and Unambiguousizability (over \mathbb{Q}) are decidable in **2-EXPTIME** (Benalioua–Lhote–Reynier 2024).
2. If the input WFA is polynomially ambiguous, then determinizability and unambiguousizability are decidable in **PSPACE** (Jecker–Mazowiecki–Purser 2023).
3. Observation: Deciding unambiguousizability or determinizability is at least as hard as the finiteness problem for matrix semigroups (? Purser).

Beyond Unambigualizability

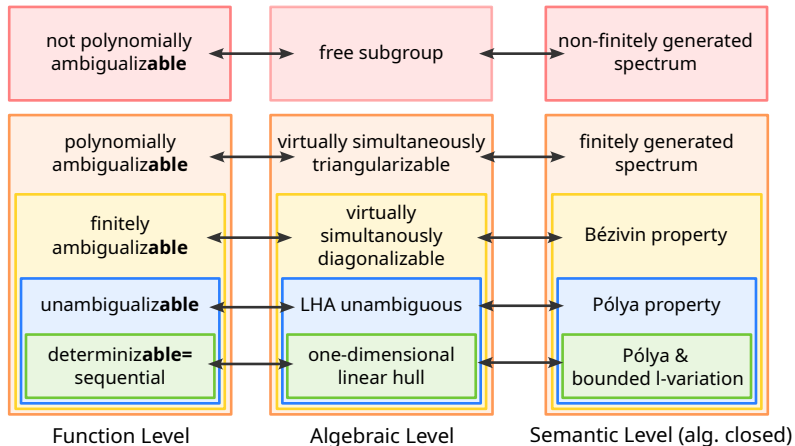
Puch-S. 2024:



- ▶ **Bézivin property:** output of a finitely ambiguable WFA are contained in $M \cdot \Gamma_0 = \Gamma_0 + \dots + \Gamma_0$ for some $M \in \mathbb{N}$ and $\Gamma \leq K^\times$ finitely generated.
- ▶ **Finitely Generated Spectrum:** there is a finitely generated $G \leq K^\times$ containing all eigenvalues of all transition matrices. (Characterization via **Pumping Sequence Families** possible, Mazowiecki–Puch–S. 2026).

The Invertible Case

Status - Invertible Case



- ▶ Semantic and Algebraic Characterization, Decidability (Puch-S. 2024).
- ▶ **Work in Progress:** Decidability in **PTIME** (Mazowiecki-S. using Beals 1998) → Tutorial.

Finitely Ambiguable WFA

Theorem (Puch-S. 2024)

Let (u, μ, v) be a minimal linear representation of f and suppose $\mu(X^*) \subseteq GL_d(K)$.

TFAE:

1. (**Automata**) f is recognized by a finitely ambiguous WFA.
2. (**Semantic**) f is Bézivin.
3. (**Algebraic I**) the group generated by $\mu(X^*)$ has a simultaneously diagonalizable finite-index subgroup (over K).
4. (**Algebraic II**) f has a representation (u', μ', v') with $\mu'(X^*)$ consisting of monomial matrices.

Remark

- ▶ Unit equations (from Diophantine number theory) are used.
- ▶ Slightly weaker result in characteristic 0 recently found by [Corvaja–Demeio–Rapinchuk–Ren–Zannier 2023](#) in studying bounded generation property.

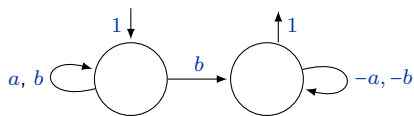
Polynomially Ambiguable WFA

Theorem (Puch-S. 2024)

Let (u, μ, v) be a minimal linear representation of f and suppose $\mu(X^*) \subseteq \text{GL}_d(K)$.
TFAE:

1. (**Automata**) f is recognized by a polynomially ambiguous WFA.
2. ("**Semantic**") f has finitely generated spectrum and K is power-splitting for $\mu(X^*)$ (for every eigenvalue some power is in K).
3. (**Algebraic I**) the group generated by $\mu(X^*)$ has a simultaneously triangularizable finite-index subgroup (over K).
4. (**Algebraic II**) f has a representation (u', μ', v') with $\mu'(X^*)$ consisting of block-triangular matrices with monomial diagonal blocks.
5. (**Algebraic III**) The group generated by $\mu(X^*)$ is virtually solvable and K is power-splitting for it.

An Example



$$u = (1 \ 0),$$
$$A = \mu(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \mu(b) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$
$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Here $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, so $(AB)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $(BA)^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$,
and

$$\text{gr}\langle A, B \rangle = N \cup AN \cong D_\infty,$$

where

$$N = \text{gr}\langle AB \rangle = \{(AB)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Does not have any diagonalizable finite-index subgroup, so the WFA is not finitely ambigulizable!

Special Case: One-Letter Alphabets (LRS)

Let $|X| = 1$, so f is a linear recurrence sequence (LRS).

Let A be the matrix of the minimal linear representation of f .

We can put A in Jordan normal form (over \overline{K}).

Corollary

1. A is polynomially ambiguable if and only if some power of each eigenvalue is in K .
2. A is finitely ambiguable if and only if A^n is diagonalizable over K for some n (meaning, A is diagonalizable in \overline{K} and some power of each nonzero eigenvalue is in K).

Example

Example (Fibonacci numbers)

$f(a^n) = F_n$: minimal matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalues $\phi_{\pm} = \frac{1 \pm \sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$.

- ▶ Over \mathbb{Q} : no power of ϕ lies in \mathbb{Q} , so f is **exponentially ambiguous**.
- ▶ Over $\mathbb{Q}(\sqrt{5})$: A is diagonalizable with eigenvalues in $\mathbb{Q}(\sqrt{5})$, so f is **finitely ambiguous** (2-ambiguous), this is just the formula

$$F_n = \frac{1}{\sqrt{5}}\phi_+^n - \frac{1}{\sqrt{5}}\phi_-^n.$$

Decidability in PTIME (Invertible Case)

- ▶ Computing the linear hull is not good, can be exponential in the input size!
- ▶ In the invertible case, only the component containing the identity matters.

Deciding virtual solvability (Tits' alternative) is exactly asking whether \mathcal{A} polynomially ambigulizable over $\overline{\mathbb{Q}}$?

Theorem (Computational Group Theory, [Beals 1998](#))

Virtual solvability for matrix groups over $\overline{\mathbb{Q}}$ is decidable in **PTIME**.

Uses

- ▶ Finiteness for matrix groups is in **PTIME** ([Babai–Beals–Rockmore 1997](#)).
- ▶ Wedderburn-Artin decomposition over \mathbb{Q} is computable in **PTIME** ([Friedl–Rónyai 1985](#)), uses that factoring polynomials over \mathbb{Q} is in **PTIME** (LLL-algorithm, [Lenstra–Lenstra–Lovász 1982](#)).

Same works for finitely ambigulizable, determinizable (\Leftrightarrow unambigulizable).

Open Problems

1. Characterize finite and polynomial ambigulizability of WFA in the non-invertible case (semantically and algebraically).
2. What is the complexity of determinizability of WFA over \mathbb{Q} in the non-invertible case?
3. Suppose \mathcal{A} is determinizable. What is the size of a minimal equivalent unambiguous or finitely ambiguous WFA?