Nonlinear extensions of linear recurrence sequences

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Based on works with:

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Linear recurrence sequences (LRS)

$a : \mathbb{N} \to \mathbb{Q}$

For example $a_n = n^2$

$$a_0 = 0, \quad a_{n+1} = a_n + 2b_n + c_n$$

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Can be also defined by $a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$

In general $u_{n+k} = \sum_{i=0}^{k-1} \alpha_i u_{n+i}$

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  e.g. $F_n = \frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}}\psi^n$

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  We work over $\mathbb{Q}$

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If P_i are rational functions this is a larger class called ratrec

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Present in the logic framework, replace quantifiers with aggregators

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Nonlinear extensions summary

LRS

(weighted automata)

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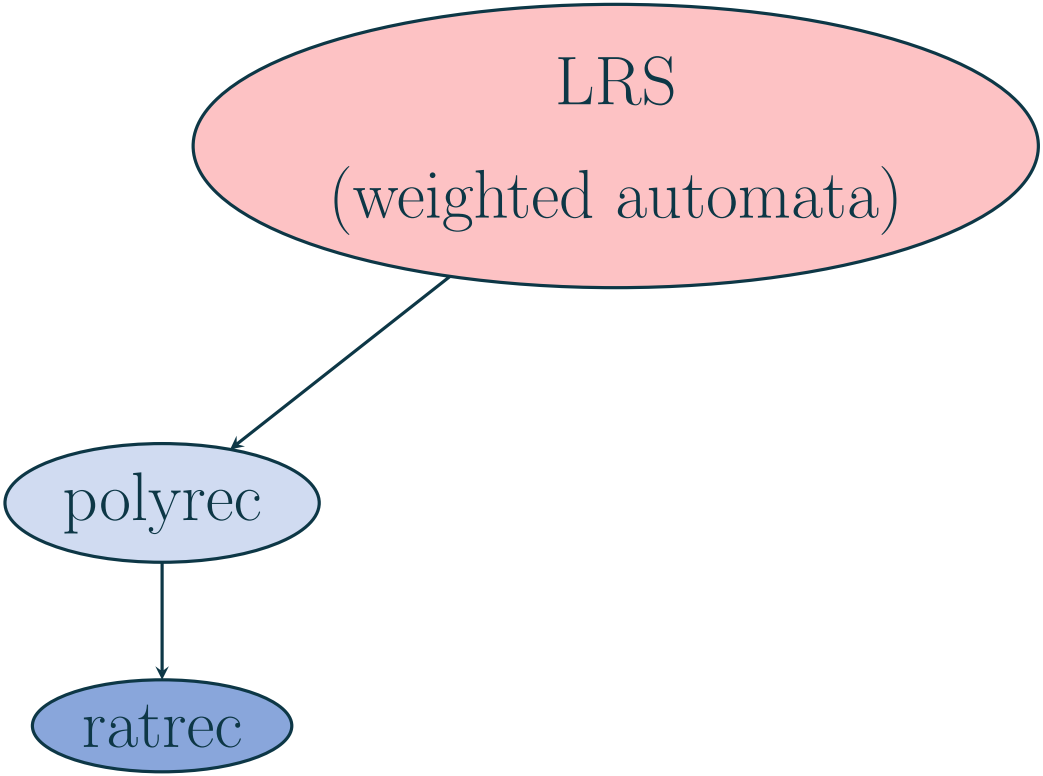

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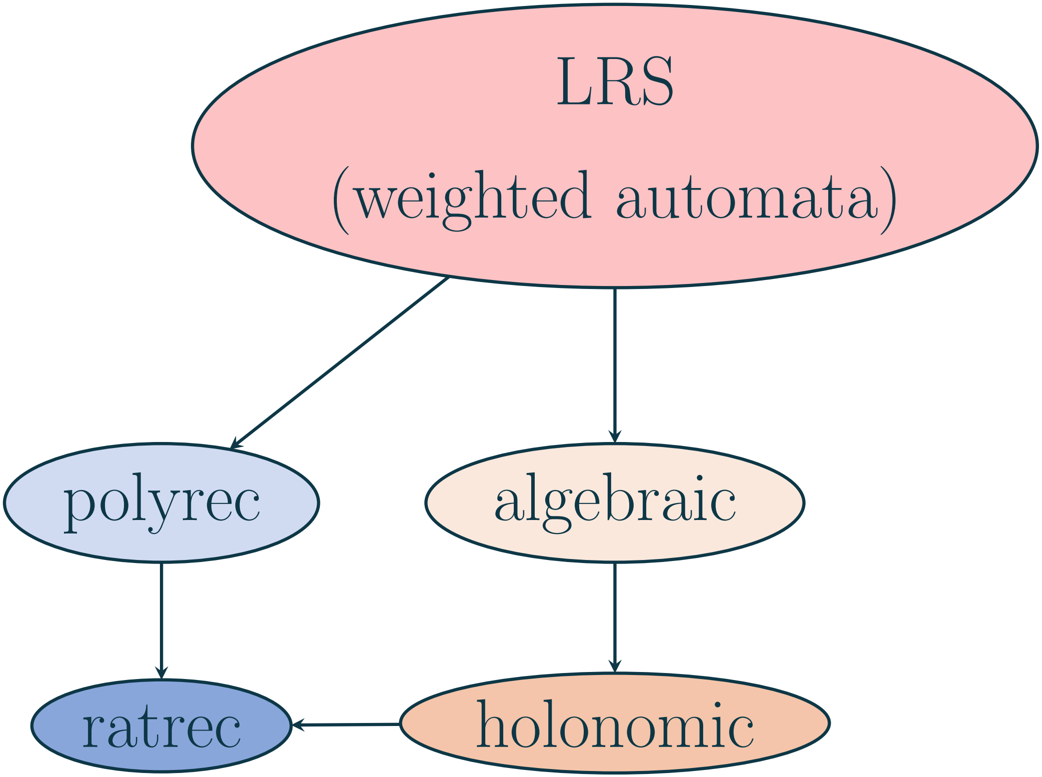

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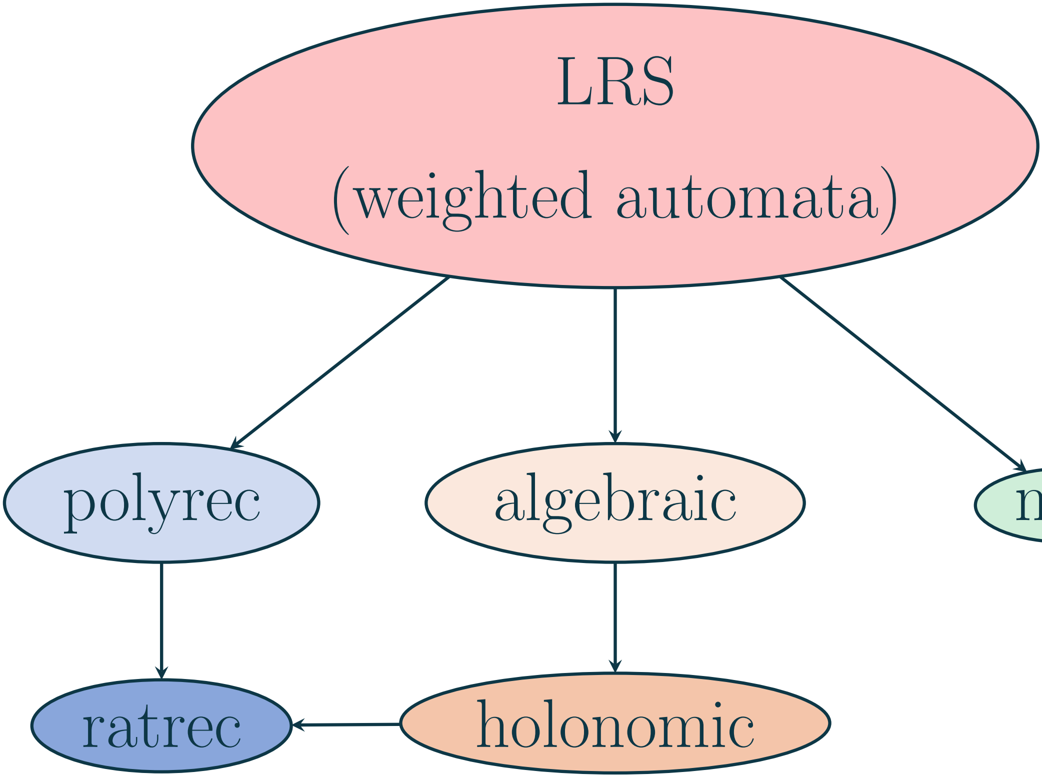

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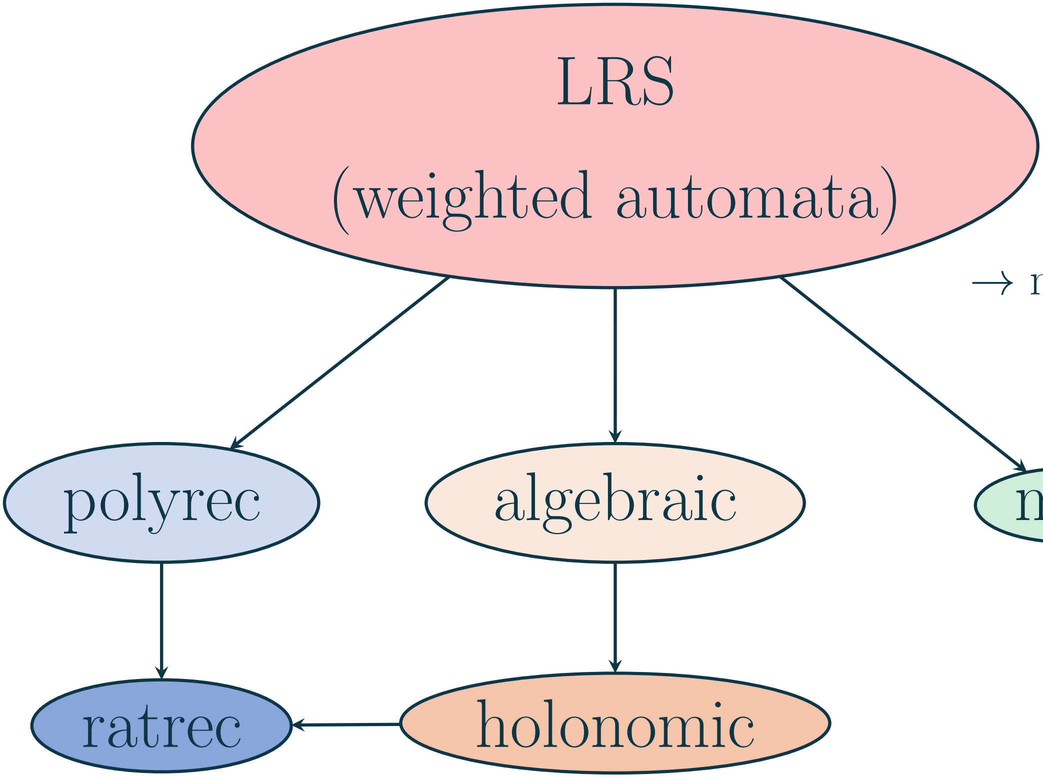

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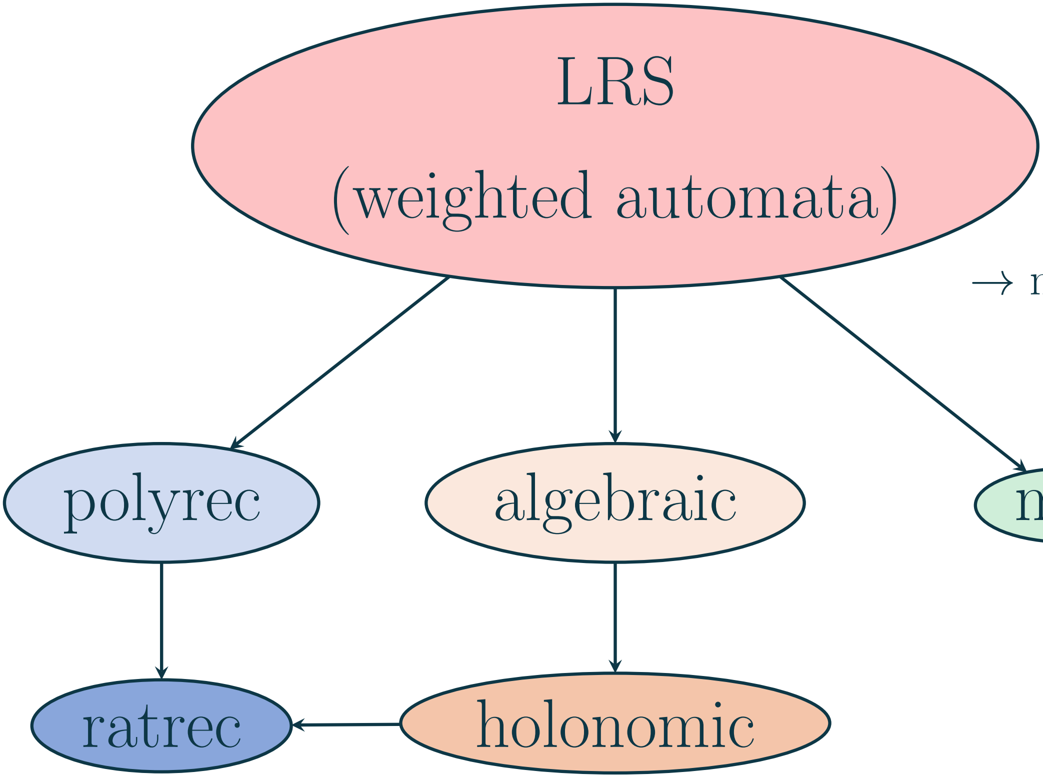

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Goal: show that there are no other inclusions.

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Also algebraic are strictly contained in holonomic

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This makes sense for p big enough ($a^i_n \in \mathbb{Q}$)

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It remains to compare polyrec/ratrec with MSO sequences

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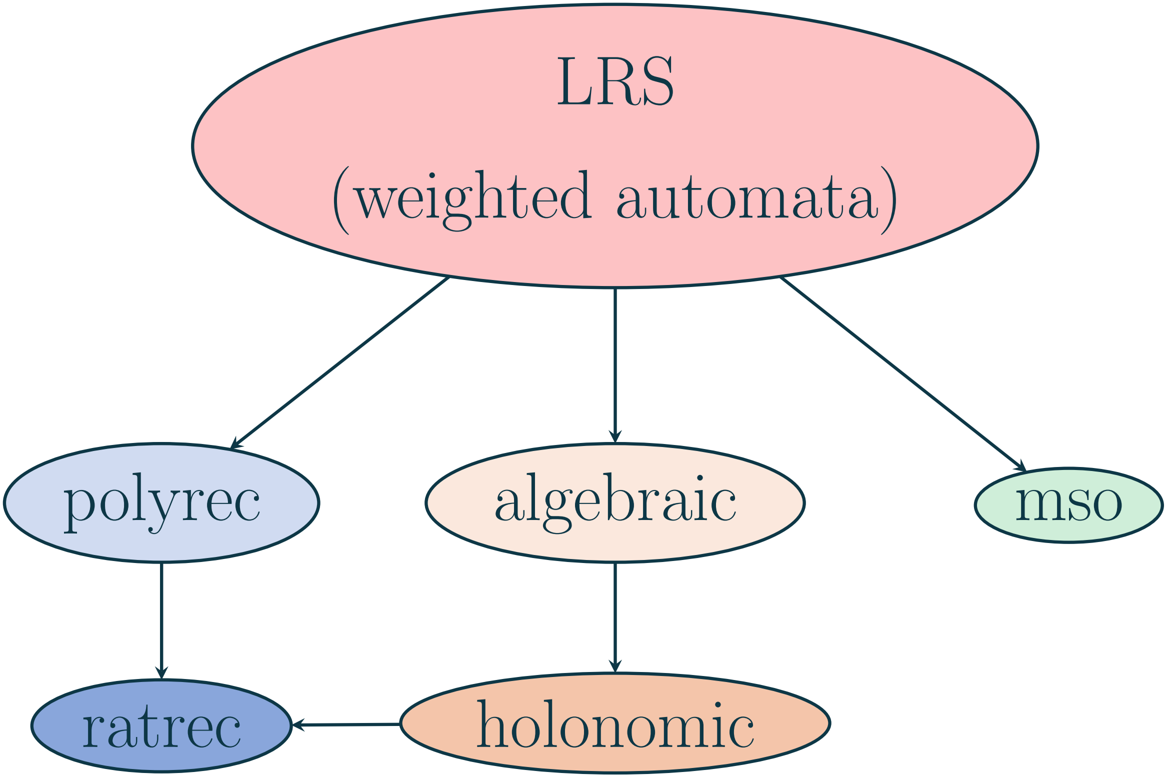

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  $\equiv n^n \cdot n^{p(p-1)} \equiv n^n \mod p$

Cancelling polynomials

Definition

A sequence a_n admits a cancelling polynomial if there exists $k \in \mathbb{N}$ and nonzero $Z \in \mathbb{Q}[x_0, \ldots, x_k]$ such that $Z(a_n, a_{n+1}, \ldots, a_{n+k}) = 0$ for all n.

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Take $k = 2$ and $Z(x_0, x_1, x_2) = x_2 \cdot x_0 - (x_1)^2 - x_1 \cdot x_0$

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Theorem

Polyrec and ratrec sequences admit cancelling polynomials

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Corollary: polyrec and ratrec do not contain MSO sequences

A polyrec sequence which is not an MSO sequence

Well, we don't know.

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Conjecture

$a_n = F_{F_n}$ is not an MSO sequence, where F_n is Fibonacci

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Maybe that double exponential comes from $\prod_X$

Composing LRS over $\mathbb{N}$

Theorem (Sénizergues)

If a_n, b_n are LRS $\mathbb{N} \to \mathbb{N}$ then $c_n = a_{b_n}$ is polyrec

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To express this we use 8 sequences

a_n, b_n, c_n, d_n (Fibonacci), x_n, y_n, z_n, w_n (previous Fibonacci)

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Note that we can compose only once (asymptotics)

Two definitions of sequences

For LRS

1. $a_{n+k} = \sum_{i=0}^{k-1} \alpha_i a_{n+i}$ (one sequence)

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Conversely see part 2

One sequence definition for polyrec?

$a_{n+k} = P(a_n, a_{n+1}, \ldots, a_{n+k-1})$ for some $P \in \mathbb{Q}[x_0, \ldots, x_{k-1}]$

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Asymptotic reasons (next slide)

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Thus $Q = 0$ (contradiction)

What about ratrec definable by one polynomial?

$a_{n+k} = R(a_n, a_{n+1}, \ldots, a_{n+k-1})$ for some $R \in \mathbb{Q}(x_0, \ldots, x_{k-1})$

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Theorem (Tabuguia and Worrell 2024)

This is true for holonomic sequences

Problematic sequence

Given a ratrec defined with d sequences

One would conjecture that the recursion depth is $k = f(d)$

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In general if $a_0, a_1, \ldots, a_{k-1}, a_k = 0$ then $a_n = 0 \quad$ (for all n)

But the first nonzero element of u_n depends on m (independent of $d = 2$)

Possible application

Checking if ratrec is well-defined is Skolem hard

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But the conjecture could be applied to polyrec

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Currently we know PSPACE-hardness and Ackermann upper bound.

Conclusion

- polyrec is interesting

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- ratrec is interesting
- Are MSO sequences interesting?

Cancelling polynomial for polyrec and ratrec

Recall LRS $a_n = I^{\top} M^n F$ (system of sequences)

Goal: $a_{n+k} = \sum_{i=0}^{k-1} \alpha_i a_{n+i}$

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Consider $R : \mathbb{Q}^k \to \mathbb{Q}^{k+1}$

$R(\vec{x}) = (I^\top M^0 \vec{x}, I^\top M^0 \vec{x}, \ldots, I^\top M^k \vec{x})$

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So there is nonzero $K : \mathbb{Q}^{n+1} \to \mathbb{Q}$ s.t. $\mathrm{Im}(R) = \ker(K)$
which defines the recursion

Cancelling polynomial for polyrec and ratrec

$$\begin{aligned} a^1_{n+1} &= P_1(a^1_n, \ldots, a^k_n) \\ a^2_{n+1} &= P_2(a^1_n, \ldots, a^k_n) \\ &\vdots \\ a^k_{n+1} &= P_k(a^1_n, \ldots, a^k_n) \end{aligned}$$

Goal nonzero $Z \in \mathbb{Q}[x_0, \ldots, x_k]$ such that $Z(a_n, a_{n+1} \ldots, a_{n+k}) = 0$ for all n

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In $\mathbb{Q}(x_1, \ldots, x_k)$ every $k+1$ elements are algebraically dependent.

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Define $P^t_i \in \mathbb{Q}[x_1, \ldots, x_k]$

$P^0_i(x_1, \ldots, x_k) = x_i, \quad$ and $P^t_i(\vec{x}) = P_i(P^{t-1}_1(\vec{x}), \ldots P^{t-1}_k(\vec{x}))$.

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Note: we can assume $R \in \mathbb{Z}[x_1, \ldots, x_k]$

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Lemma

Suppose $0 \neq Z \in \mathbb{Z}[x_0, \ldots, x_k]$. There exists $P_1, \ldots, P_m, Q_1, \ldots, Q_m \in \mathbb{Z}[x]$ s.t. $Z(n^n, (n+1)^{n+1}, \ldots, (n+k)^{n+k}) = \sum_{i=1}^{m} P_i(n)^n \cdot Q_i(n)$ for all n.

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Observe $Q_i(n) \equiv Q_i(a) \mod p$, and $P_i(n)^n \equiv P_i(a)^b \mod p$

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$\vec{u}_a = (Q_1(a), \ldots, Q_m(a))$

$$D_a = \begin{pmatrix} P_1(a) & P_1(a)^2 & \ldots & P_1(a)^m \\ P_2(a) & P_2(a)^2 & \ldots & P_2(a)^m \\ & & \vdots & \\ P_m(a) & P_m(a)^2 & \ldots & P_m(a)^m \end{pmatrix}$$

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(Almost) Vandermonde matrix

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$$\det(D_a) = \prod_{i=1}^{m} P_i(a) \cdot \prod_{1 \leq i < j \leq m} (P_i(a) - P_j(a))$$

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Take big p

and recall that nonzero polynomials have bounded number of zeros