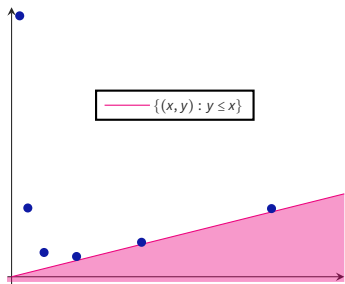


POSITIVITY PROBLEMS AND EQUALITY TESTS



George Kenison

POSITIVITY AND THE HALTING PROBLEM



Does the following loop halt?

$$(x, y) = (2^{-10}, 1);$$

while $y \geq x$ **do**

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

end while

i.e., do we have

$$M^n \alpha \in \{(x, y) : y \geq x\}, \forall n \in \mathbb{N}?$$

The ‘famous’ halfplane version is commonly phrased in terms of LRSs.

POSITIVITY

Given an integer LRS $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.

DECISION PROBLEMS FOR RECURRENCE SEQUENCES

(Are these problems decidable?) [OW15]

Given an LRS $\langle u_n \rangle_{n=0}^{\infty}$ (over \mathbb{Q} or \mathbb{Z})

- Does $u_n = 0$ for some n ? **SKOLEM** is open.
- Does $u_n = 0$ for infinitely many n ? is decidable (Berstel–Mignotte)
- Is $u_n \geq 0$ for all n ? **POSITIVITY** is open.
- Is $u_n \geq 0$ for all but finitely many n ? **ULTIMATE POSITIVITY** is open.

An LRS satisfies a rec. relation $u_n = a_1 u_{n-1} + \dots + a_k u_{n-k}$ with initial values u_0, \dots, u_{k-1}

POSITIVITY RECAP [OW14]

POSITIVITY

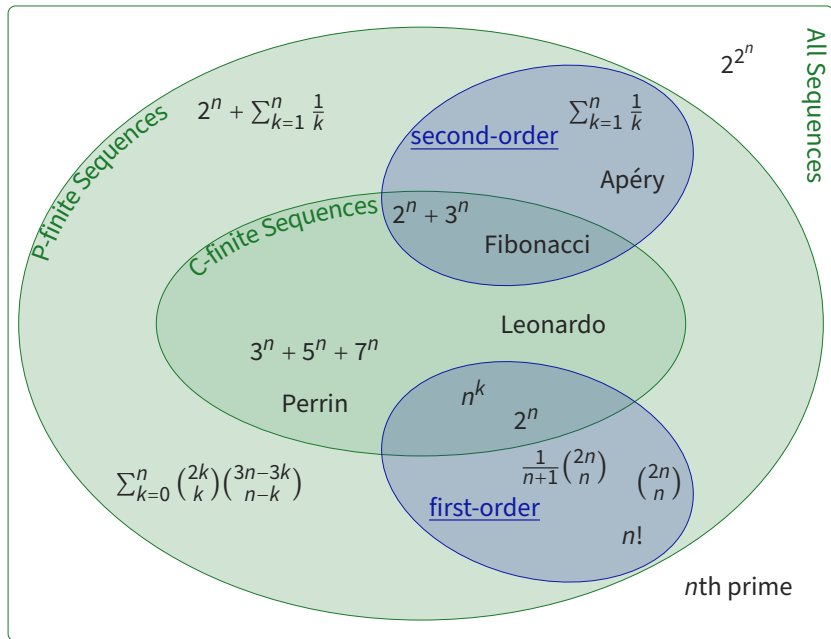
Given a recursively defined seq $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.

For LRSs (C-finite),

- POSITIVITY is decidable up to order 5.
- If POSITIVITY is decidable at order 6, then we can compute Lagrange constants for a large class of transcendental numbers.

How about a different family of recursively defined sequences?

THE LANDSCAPE OF P-FINITE SEQUENCES [KP11]



POSITIVITY FOR SECOND-ORDER P-FINITE SEQUENCES

- [Ken+] G. Kenison, O. Klurman, E. Lefauchaux, F. Luca, P. Moree, E. C. Sertöz, J. Ouaknine, A. Whiteland, and J. Worrell. *On the Positivity Problem for second-order holonomic sequences*.
- [Ken+21] G. Kenison, O. Klurman, E. Lefauchaux, F. Luca, P. Moree, J. Ouaknine, M. A. Whiteland, and J. Worrell. “On Positivity and Minimality for Second-Order Holonomic Sequences”. In: *International Symposium on Mathematical Foundations of Computer Science, MFCS 2021*. 2021, 67:1–67:15.

Many more papers (e.g., [HK25; IS24; Ken24; KP10]) consider POSITIVITY and related problems in the P-finite (not necessarily second-order) setting.

SECOND-ORDER P-FINITE SEQUENCES

rational sequences $\langle u_n \rangle_{n=-1}^{\infty}$ that satisfy relations of the form

$$u_n = b_n u_{n-1} + a_n u_{n-2} \quad \text{for } n = 1, 2, 3, \dots$$

with $a_n := \frac{p_1(n)}{p_3(n)}$ and $b_n := \frac{p_2(n)}{p_3(n)}$.

The solution space of the recurrence relation is a 2-dim vector space

- A soln $\langle u_n \rangle_n$ is uniquely determined by the pair u_{-1} and u_0 .
- $\langle 0 \rangle_n$ is a soln.
- if $\langle u_n \rangle_n$ is a soln, then so is $\alpha \langle u_n \rangle_n = \langle \alpha u_n \rangle_n$ for every $\alpha \in \mathbb{R}$.
- if $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ are solutions, then so is $\langle w_n \rangle_n + \langle v_n \rangle_n = \langle v_n + w_n \rangle_n$.

SOLUTION SPACE AND RECURRENCE SEQUENCE PROBLEMS

Definition (Minimal solutions)

A soln $\langle u_n \rangle_n$ to $p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$ is *minimal* if $\langle u_n \rangle_n$ is non-trivial and there exists a second soln $\langle v_n \rangle_n$ such that $\lim_{n \rightarrow \infty} u_n/v_n = 0$.

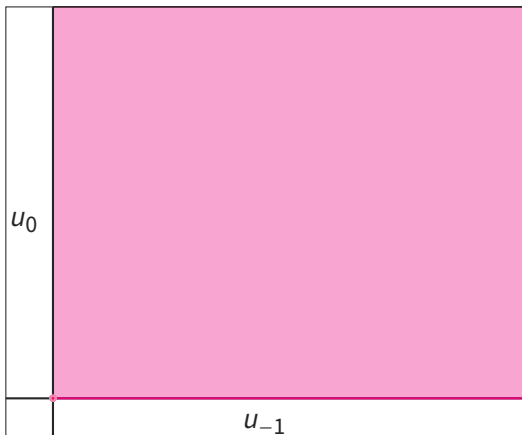
Decision Problems for Recurrence Relations (Solution Space)

- Does a given recurrence relation admit Positive solutions? **decidable**
- Does a given recurrence relation admit Minimal solutions? **decidable**

Decision Problems for Recurrence Sequences

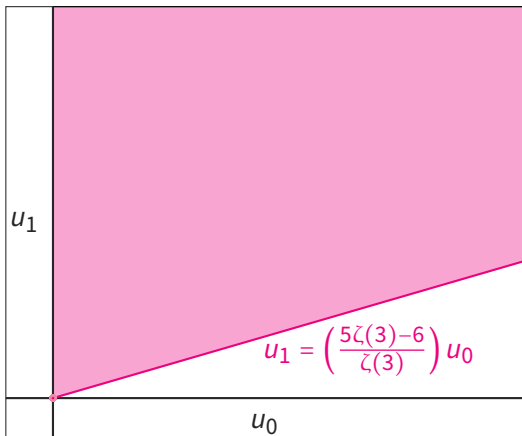
- (POSITIVITY) Is a given sequence $\langle u_n \rangle_n$ Positive? (i.e., $u_n \geq 0, \forall n$)
- (MINIMALITY) Is a given sequence $\langle u_n \rangle_n$ Minimal?

POSITIVE SOLUTIONS TO $u_{n+1} = u_n + u_{n-1}$



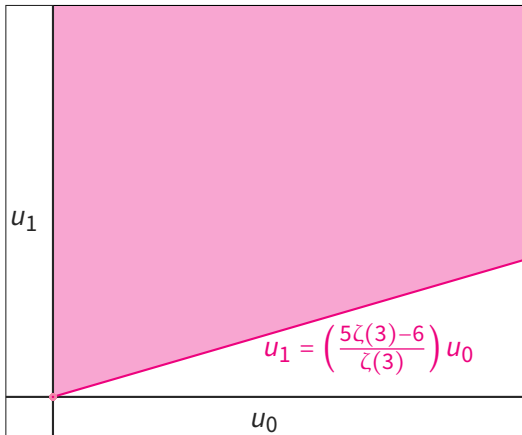
Set $I = \{(u_{-1}, u_0) : u_n \geq 0, \forall n\}$ of initial values that determine a positive solution are closed under linear combinations with positive coefficients, so I is a convex linear cone.

POSITIVE SOLUTIONS TO $n^3 u_{n+1} = (34n^3 - 51n^2 + 27n - 5)u_n - (n-1)^3 u_{n-1}$



Set $I = \{(u_0, u_1) : u_n \geq 0, \forall n\}$ is still a convex linear cone.

POSITIVE SOLUTIONS TO $n^3 u_{n+1} = (34n^3 - 51n^2 + 27n - 5)u_n - (n-1)^3 u_{n-1}$



$$\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, \dots \rangle, \quad \langle B_n \rangle_{n=0}^\infty = \langle 0, 6, 351/4, 62531/36, \dots \rangle.$$

Because $\frac{\zeta(3)A_n - B_n}{A_n} \rightarrow 0$ [Apé79; Poo79], $\alpha \langle \zeta(3)A_n - B_n \rangle_n$ is a *minimal* soln (substitute initial values!).

THE CONTINUED FRACTION CONNECTION

$$\mathbf{K}_{k=1}^n \frac{a_k}{b_k} := \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ddots + \frac{b_n}{a_n}}}}$$

Theorem (Pincherle 1894 [Pin94])

- $\mathbf{K}(a_k/b_k)$ converges in $\hat{\mathbb{R}}$ if and only if $u_n = b_n u_{n-1} + a_n u_{n-2}$ admits minimal solutions.
- If $\langle w_n \rangle_{n=-1}^{\infty}$ is a minimal solution of $u_n = b_n u_{n-1} + a_n u_{n-2}$ then $\mathbf{K}(a_k/b_k)$ converges to $-w_0/w_{-1} \in \hat{\mathbb{R}}$.

THE CONTINUED FRACTION CONNECTION

$$\mathbf{K}_{k=1}^n \frac{a_k}{b_k} := \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ddots + \frac{b_n}{a_n}}}}$$

$$\phi = 1 + \mathbf{K}_{k=1}^{\infty} \frac{1}{1},$$

$$\frac{4}{\pi} = 1 + \mathbf{K}_{k=1}^{\infty} \frac{(2k-1)^2}{2},$$

$$\frac{5\zeta(3) - 6}{\zeta(3)} = \mathbf{K}_{k=1}^{\infty} \frac{-k^6}{34k^3 + 51k^2 + 27k + 5}.$$

Theorem ([Ken+21])

Given a second-order P-finite recurrence relation

$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$. We have the following:

1. There is a convex linear cone such that all terms of a solution $\langle u_n \rangle_n$ are non-negative if and only if u_0/u_{-1} lies in the cone.
2. We can decide POSITIVITY except when u_0/u_{-1} lies on the boundary of the cone.

Further, for the class of second-order P-finite sequences, the problem of deciding POSITIVITY Turing-reduces to that of MINIMALITY (i.e., a procedure to decide MINIMALITY implies the existence of a procedure to decide POSITIVITY).

REFERENCES [1]

- [Apé79] R. Apéry. “Irrationalité de $\zeta(2)$ et $\zeta(3)$ ”. In: *Astérisque* 61 (1979). Luminy Conference on Arithmetic, pp. 11–13.
- [HK25] F. Hagihara and A. Kawamura. “The Ultimate Signs of Second-Order Holonomic Sequences”. In: *52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025)*. Vol. 334. 2025, 159:1–159:20.
- [IS24] A. Ibrahim and B. Salvy. “Positivity Certificates for Linear Recurrences”. In: *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2024, pp. 982–994. eprint: <https://epubs.siam.org/doi/pdf/10.1137/1.9781611977912.37>.
- [Ken+] G. Kenison, O. Klurman, E. Lefauchaux, F. Luca, P. Moree, E. C. Sertöz, J. Ouaknine, A. Whiteland, and J. Worrell. *On the Positivity Problem for second-order holonomic sequences*.

REFERENCES [2]

- [Ken+21] G. Kenison, O. Klurman, E. Lefauchaux, F. Luca, P. Moree, J. Ouaknine, M. A. Whiteland, and J. Worrell. “On Positivity and Minimality for Second-Order Holonomic Sequences”. In: *International Symposium on Mathematical Foundations of Computer Science, MFCS 2021*. 2021, 67:1–67:15.
- [Ken24] G. Kenison. “The Threshold Problem for Hypergeometric Sequences with Quadratic Parameters”. In: *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*. Vol. 297. 2024, 145:1–145:20.
- [KP10] M. Kauers and V. Pillwein. “When can we detect that a P-finite sequence is positive?” In: *Symbolic and Algebraic Computation, International Symposium, ISSAC 2010, Munich, Germany, July 25-28, 2010, Proceedings*. 2010, pp. 195–201.
- [KP11] M. Kauers and P. Paule. *The concrete tetrahedron*. Symbolic sums, recurrence equations, generating functions, asymptotic estimates. SpringerWienNewYork, Vienna, 2011, pp. x+203.

REFERENCES [3]

- [OW14] J. Ouaknine and J. Worrell. “Positivity problems for low-order linear recurrence sequences”. In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. 2014, pp. 366–379.
- [OW15] J. Ouaknine and J. Worrell. “On Linear Recurrence Sequences and Loop Termination”. In: *ACM SIGLOG News* 2.2 (2015), pp. 4–13.
- [Pin94] S. Pincherle. “Delle Funzioni ipergeometriche, e di varie questioni ad esse attinenti”. In: *Giorn. Mat. Battaglini* 32 (1894), pp. 209–291.
- [Poo79] A. van der Poorten. “A proof that Euler missed ...: An informal report”. In: *The Mathematical Intelligencer* 1.4 (1979), pp. 195–203.

Sequences obeying relations of the form

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$$

are the *second-order degree-one P-finite sequences*.

Theorem ([Ken+])

POSITIVITY and MINIMALITY are decidable for the class of second-order degree-one P-finite sequences with two distinct characteristic roots (i.e., associated $p(x) = \alpha_1 x^2 - \beta_1 x - \gamma_1$ has two distinct rational roots).

Under these restrictive conditions, the equality checks performed as part of the decision procedure make use of the following recent results:

1. **[FR22]**, for determining all polynomial relations of a given family of E-functions evaluated at a common algebraic point.
2. **[SOW]**, for determining all linear relations of 1-periods.