SAMSA-26

SYNCHRONIZING INFINITE AUTOMATA: COLLATZ PROBLEM

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I. INTRODUCTION

The Collatz function $C(n)$ is defined on the natural numbers $\mathbb{N}$ as follows:

$$C(n) = \begin{cases} n/2, & \text{if } n \text{ even} \\ 3n+1, & \text{if } n \text{ odd} \end{cases}. \tag{1}$$

Conjecture 1. For each natural number n, there is a natural number i such that $C^{(i)}(n) = 1$.

After achieving state 1, process enters to the trivial cycle (1,4,2). Thus, the trivial cycle is the attractor of dynamic system $(\mathbb{N}, C)$. The problem is open near 80 years.

n

(1,4,2)

The main difficulty here is that we have no a priori bound on the number of iterations as a function of n, so we can only check statements with experiments but not to prove it.

Let $\mathbb{N}_0$ and $\mathbb{N}_1$ be the subsets of the even and odd natural numbers, respectively.

$$\mathrm{ord}(n) = \max\{m : 2^m \text{ divides } n\} .$$

Consider the function $C_1 : \mathbb{N}_1 \to \mathbb{N}_1$, which is defined as follows:

$$C_1(n) = \frac{3n+1}{2^{\mathrm{ord}(3n+1)}} . \tag{2}$$

The state 1 becomes the unique fixed point of the function C_1. Therefore, conjecture 1 equivalent to the following statement.

Conjecture 2. The state 1 is the attractor of dynamic system $(\mathbb{N}_1, C_1)$.

This dynamic system can be represented by autonomous infinite automaton $A_1 = (\mathbb{N}_1, \{1\}, \delta_1)$, where $\delta_1(n, 1) = C_1(n)$. According conjecture 2 the super-word 1^∞ will be synchronizing in A_1.

II. Tree structure

Thus, according conjecture 2 the diagram of automaton A_1 on the set $\mathbb{N}_1$ should be a tree with 1 in the root. This is a terrible tree appears in many papers (see [2]), but its existence is equivalent to conjecture 2 (Fig. 1).

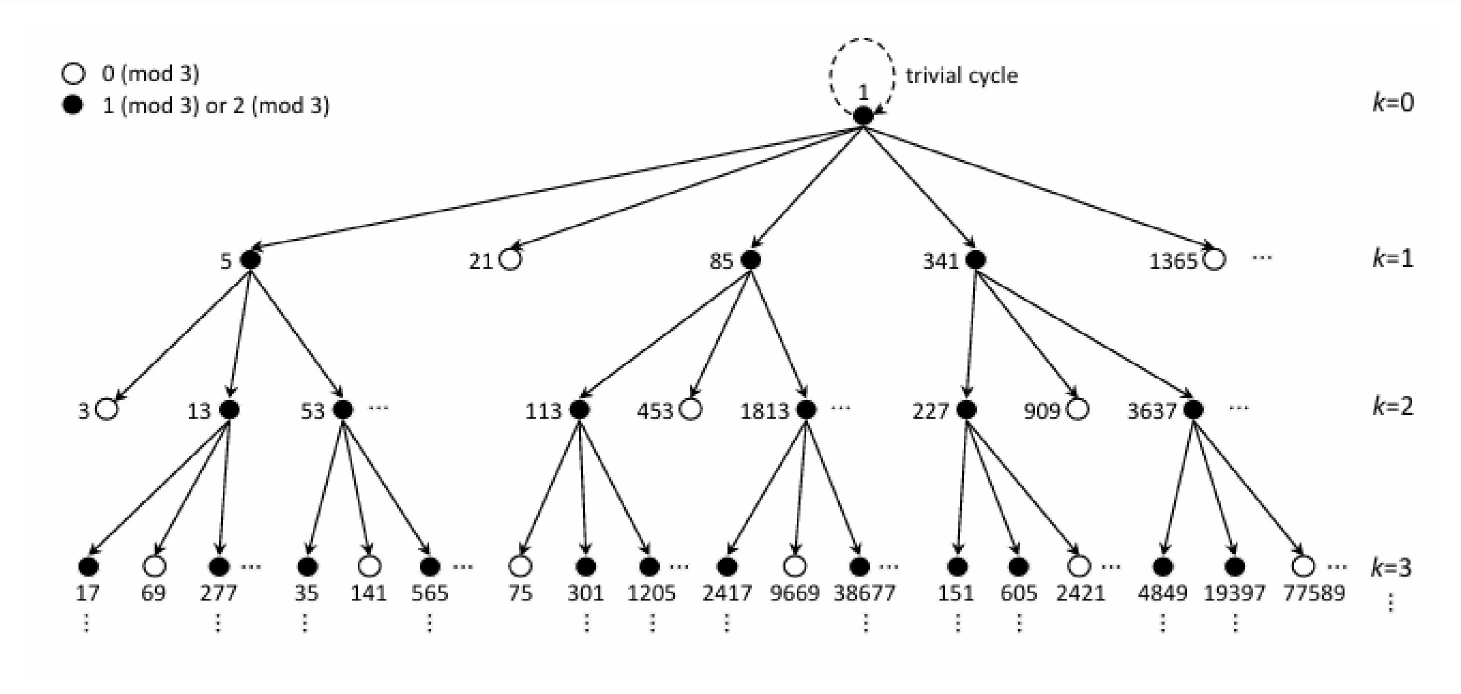

Fig. 1. Structure of automaton A_1

Proposition 1. If $n \in \mathbb{N}_1$ then $C_1(n) = C_1(4n+1)$.

Proof. We have from formula (2):

$$C_1(4n+1) = \frac{12n+4}{2^{\operatorname{ord}(12n+4)}} = \frac{4(3n+1)}{4 \cdot 2^{\operatorname{ord}(3n+1)}} = C_1(n)\,.$$

A preimage $C_1^{-1}(n) = \{k \in \mathbb{N}_1 \ : \ C_1(k) = n\}$ of a number $n \in \mathbb{N}_1$ we call a layer. From proposition 1 we have the following formula for preimages:

$$C_1^{-1}(n) = \begin{cases} \emptyset, & \text{if } n = 0 \bmod(3), \\ \left\{\dfrac{2^{2m}n-1}{3} \ : \ m \geq 1\right\}, & \text{if } n = 1 \bmod(3), \\ \left\{\dfrac{2^{2m-1}n-1}{3} : m \geq 1\right\}, & \text{if } n = 2 \bmod(3). \end{cases} \qquad (3)$$

Of course, we must point the arrows in Fig. 1 in the other direction, to talk about preimages.

III. ODD AND EVEN NUMBERS

$$\mathbb{Q}_2 = \{2^m n : m \in \mathbb{Z}, n \in \mathbb{N}_1\},$$

where $\mathbb{Z}$ is the set of integers. Note that $\mathbb{N} \subset \mathbb{Q}_2$.

If $q = 2^m n \in \mathbb{Q}_2$ then $\mathrm{ev}(q) = 2^m$ - an even part of q and respectively $\mathrm{od}(q) = n$ - an odd part. The numbers $q, r \in \mathbb{Q}_2$ is equivalent $q \sim r$ if $\mathrm{od}(q) = \mathrm{od}(r)$.

In every equivalent class $K = \langle n \rangle$, where $n \in \mathbb{N}$, there is only one odd natural number, which is called the leader of the class. Particularly, we have the first class $\langle 1 \rangle = \{2^m : m \in \mathbb{Z}\}$ with leader 1. The function:

$$C_2(q) = 3q + \mathrm{ev}(q) = 2^m(3n+1) \quad , \qquad q \in \mathbb{Q}_2 \quad , \qquad (4)$$

maps equivalent class $\langle n \rangle$, $n \in \mathbb{N}_1$ to the class $\langle 3n+1 \rangle$ in a one-to-one manner [3]. Therefore, conjectures 1 and 2 are equivalent to the statement that we can achieve the first class $C_2^{(i)}(n) \in \langle 1 \rangle$. This avoids divisions in the iterative process, but forces us to go to infinity.

IV. Cutting-off the tail

We may separate multiplication and division operations in formula (2), which is especially clear in the binary number system. Let us consider the function $C_3 : \mathbb{N}_1 \to \mathbb{N}_1$, which defined by formula:

$$C_3(n) = \frac{n+1}{2^{\operatorname{ord}(n+1)}} \quad . \tag{5}$$

This operation is called "cutting-off the tail" since the number of ones in the tail of an odd binary number n is the same as the number of zeros in the tail of an even binary number $n + 1$.

$$C_3\left(w01^k\right) = w1 \; .$$

Consider the automaton $A_2 = (\mathbb{N}_1, \{a, b\}, \delta_2)$, where:

$$\delta_2(n, a) = 3n \;, \qquad \delta_2(n, b) = C_3(n) \quad . \tag{6}$$

It is easy to see that:

$$\delta_2(n, ab) = \delta_2(3n, b) = C_3(3n) = C_1(n).$$

Thus according conjecture 2 automaton A_2 admits the synchronizing super-word $(ab)^\infty$.

Here we can see the struggle between multiplication and division.

a: $n \to 3n$, +1 or + 2 digits.

Adding one followed by division removes at least one last digit in the last block of ones in an odd binary number $3n$ and replaces the last 0 with 1:

b: $3n \to C_3(3n)$, -1 or more digits.

Oddly enough, division wins. This can be explained by the fact that the tail of any length is cut-off, while the binary number can increase by no more than one digit.

References

[1] J. C. Lagarias, The 3x+1 problem: an overview. http://arXiv: 2111.02635v1 [math.NT] 4 Nov 2021.

[2] Z. B. Batang, Integer patterns in Collatz sequences. . http://arXiv: 1907.07088v2 [math.GM] 17 Jul 2019.

[3] I. K. Rystsov, Some remarks about the Collatz problem. *Cybernetics and Systems Analysis*, vol. 49, no. 3, pp. 353-365, 2013.

Thank you for attention