Shapes of potential counterexamples to Nivat's conjecture

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– Joint work with Pierre Guillon, Jarkko Kari and Etienne Moutot –

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Nivat's conjecture is the 2D generalization of the Morse-Hedlund theorem

Pattern complexity

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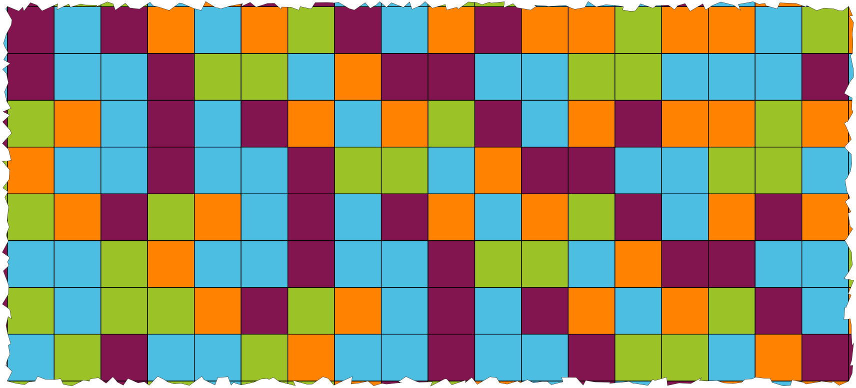

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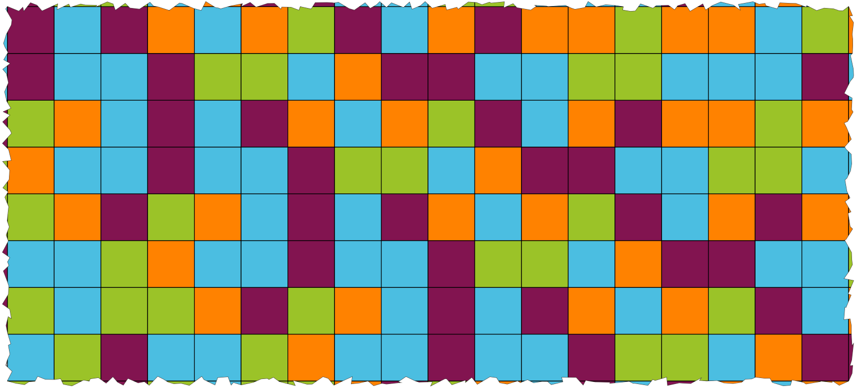

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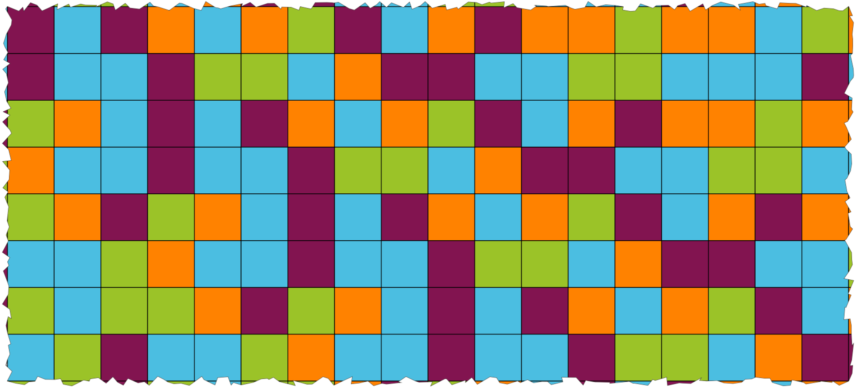

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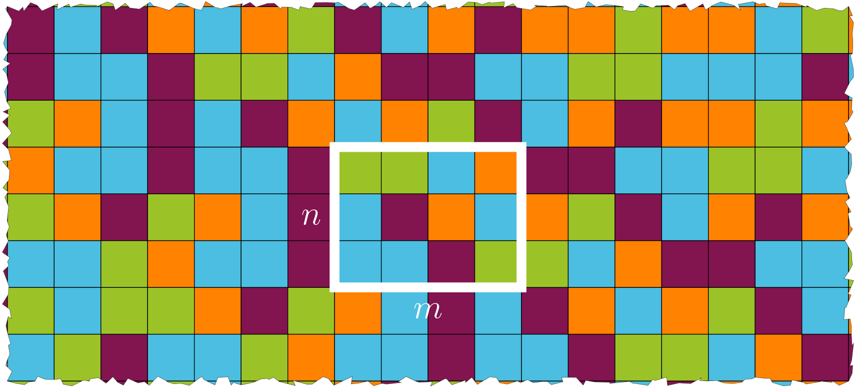

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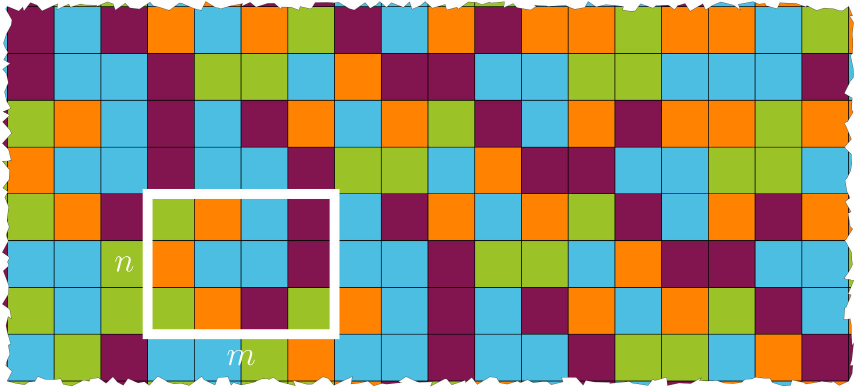

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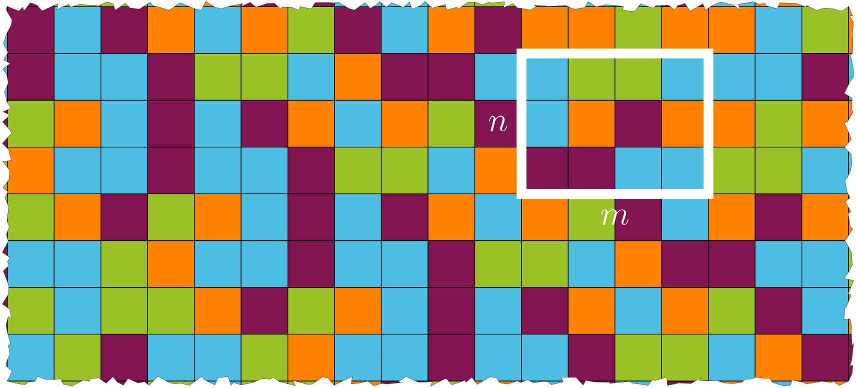

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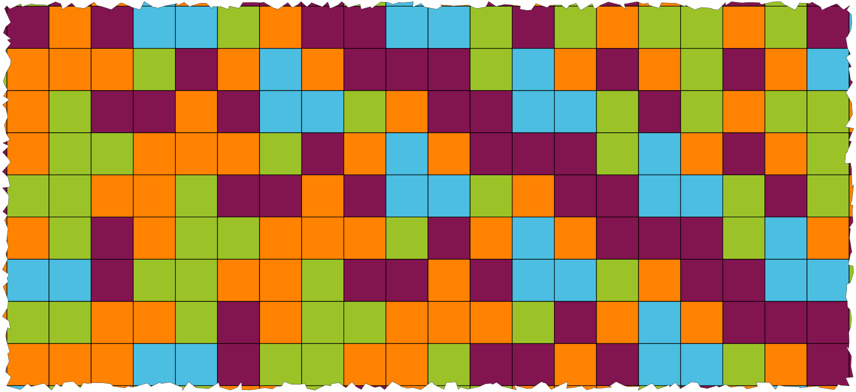

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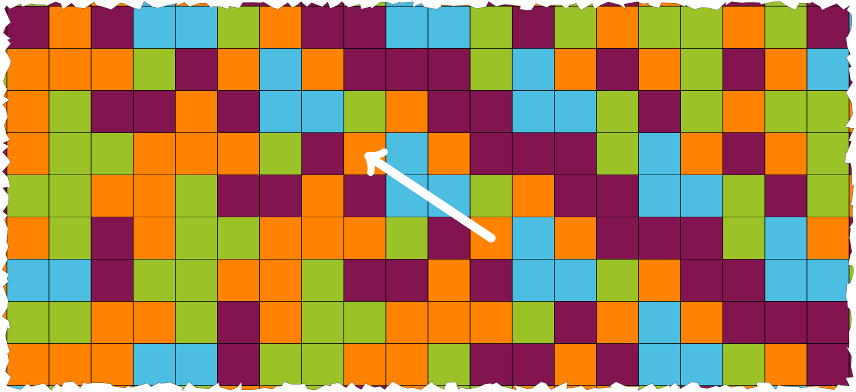

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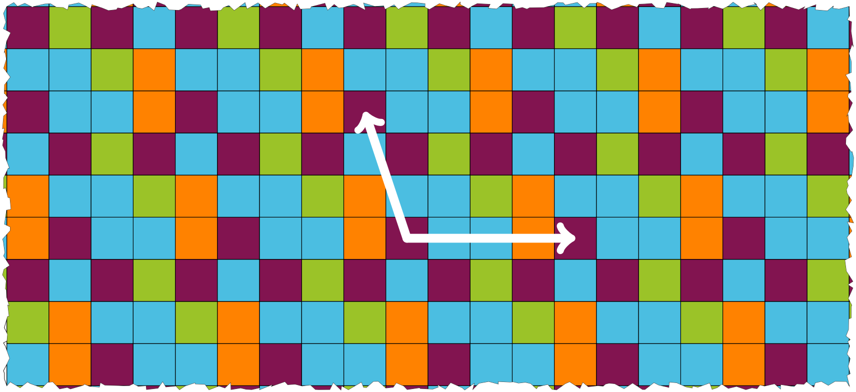

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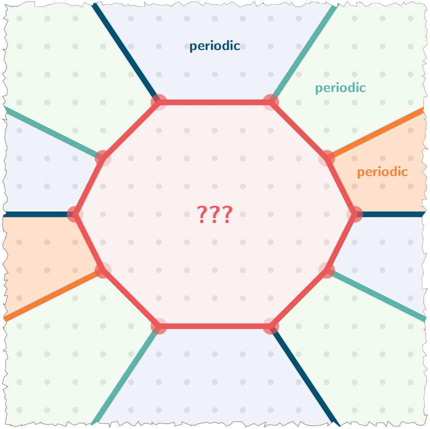

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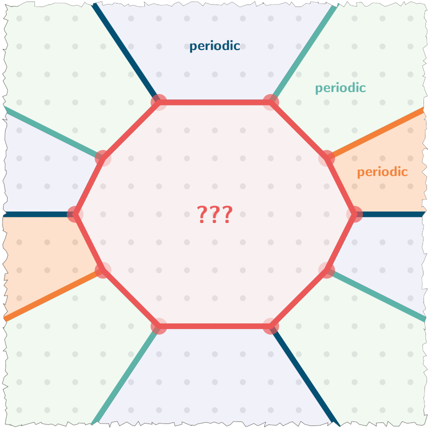

Using an algebraic approach developed by Jarkko Kari and Michal Szabados [Sza18].

An algebraic approach later...

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However, the key lemma she never managed to prove as she had constructed a counterexample to it.

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If c is a counterexample, then there exists a vector $\mathbf{v} \in \mathbb{Z}^d$ such that arbitrary large regions of "0" appear in $(X^{\mathbf{v}} - 1)c$.

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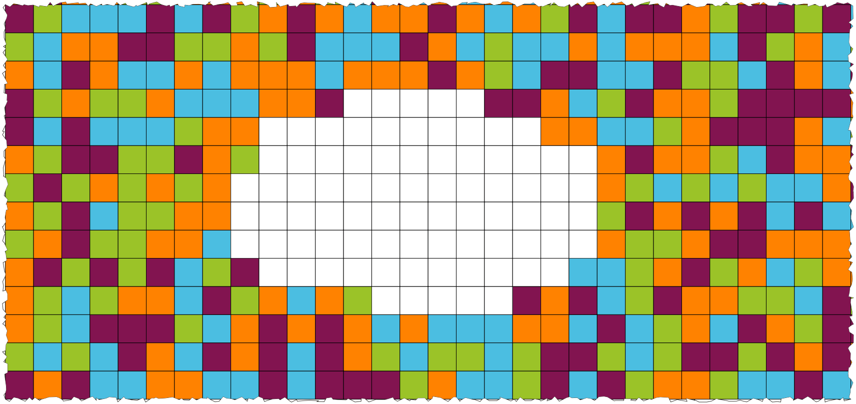

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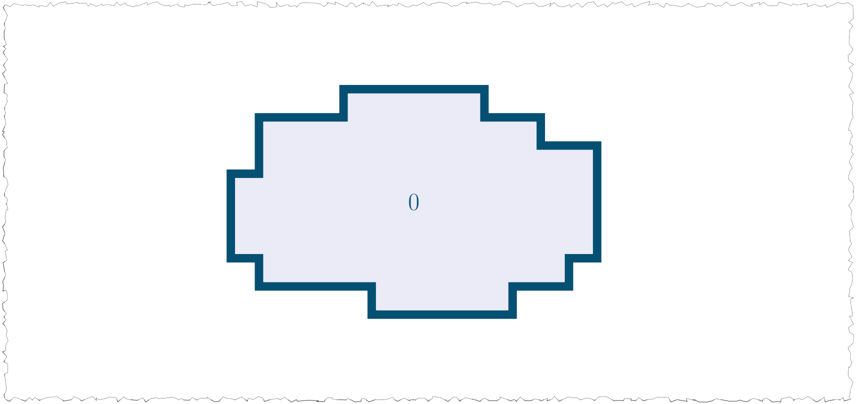

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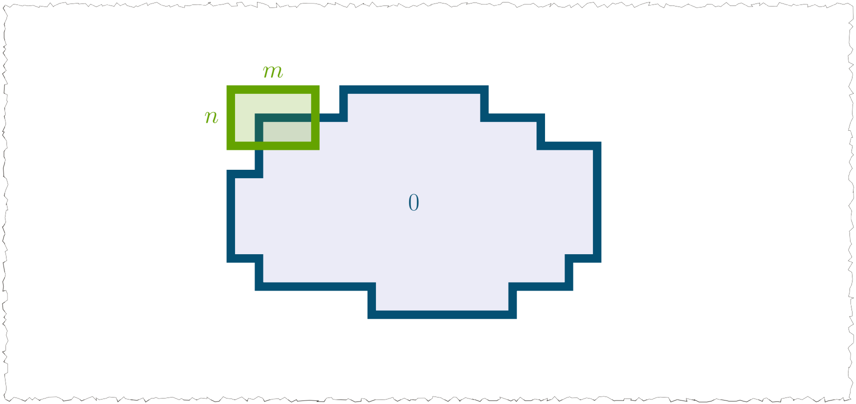

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The algebraic approach provides us with m lin. independant vectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$ and a set of points ϕ which is their "product".

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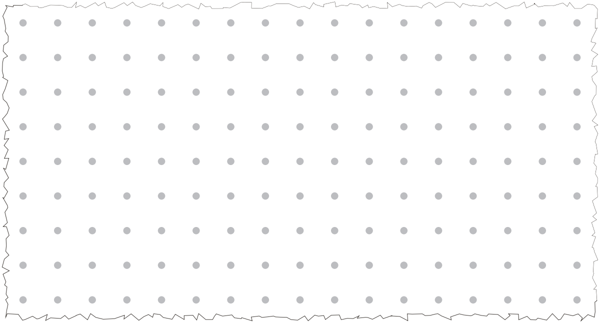

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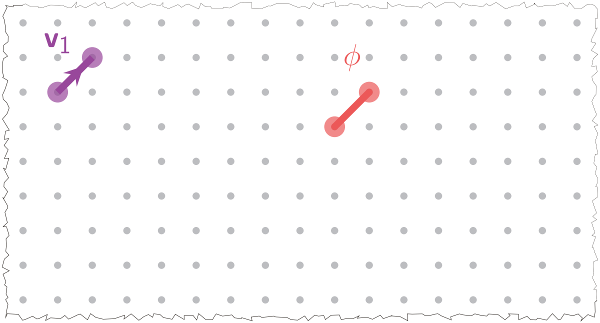

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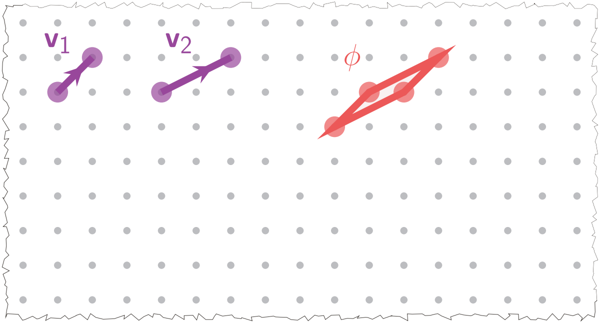

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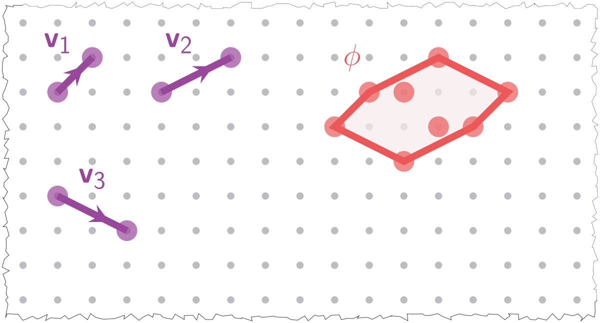

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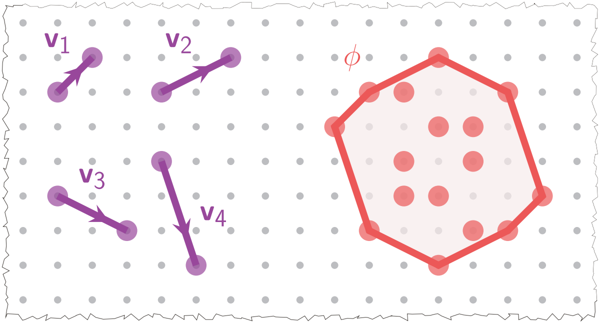

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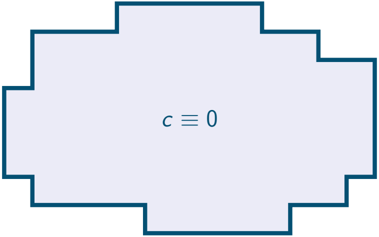

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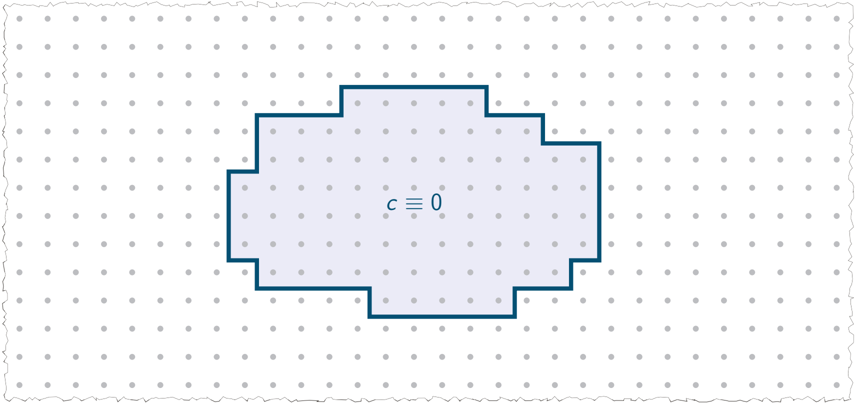

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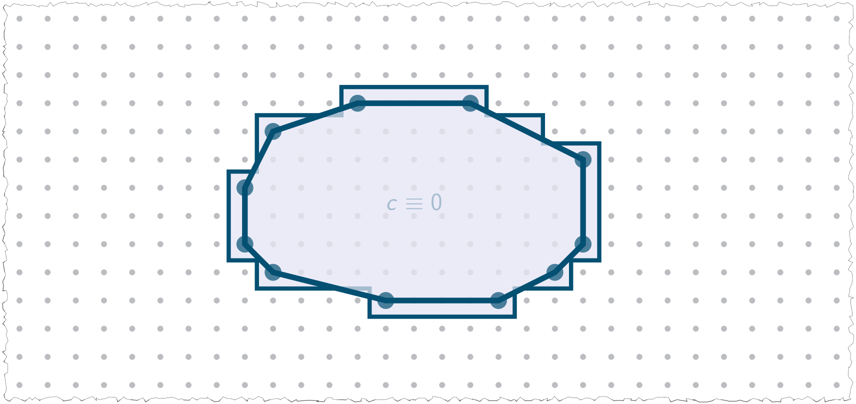

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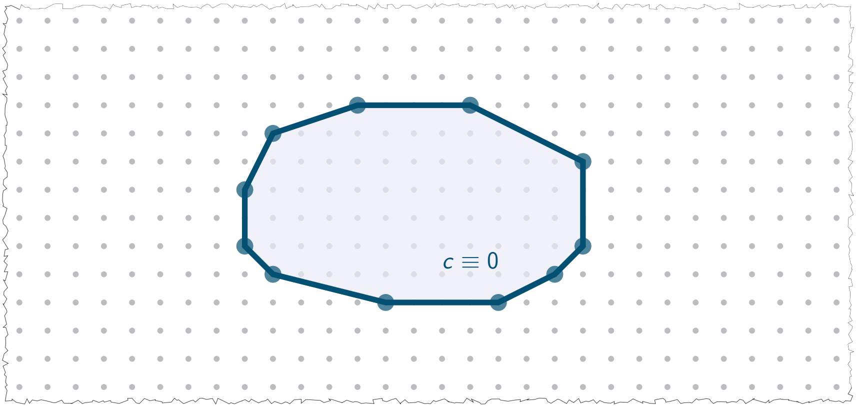

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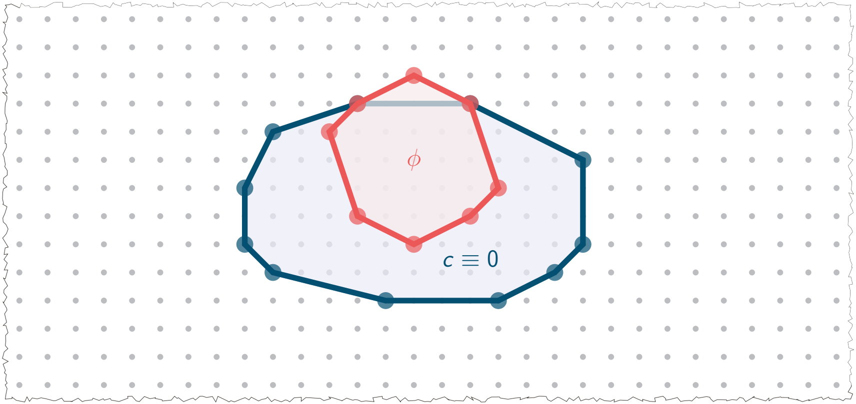

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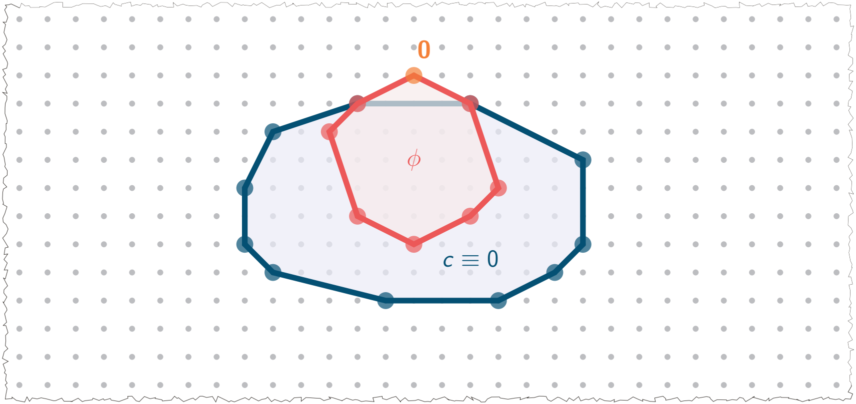

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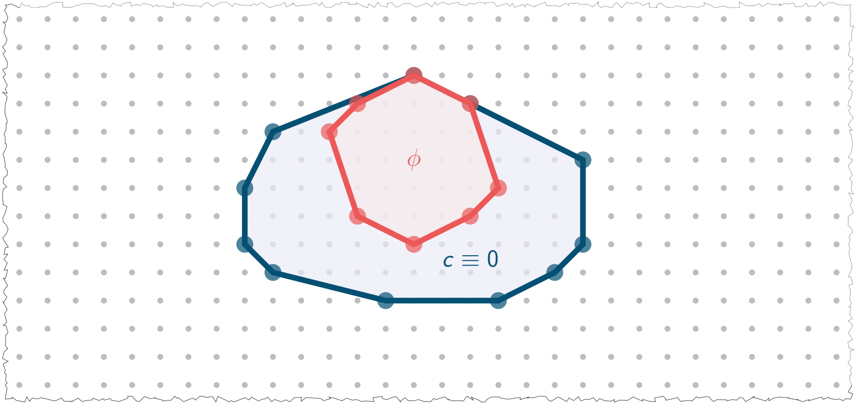
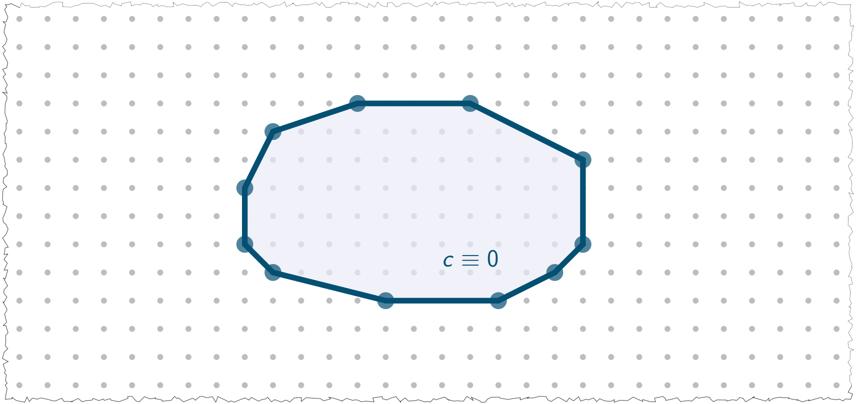
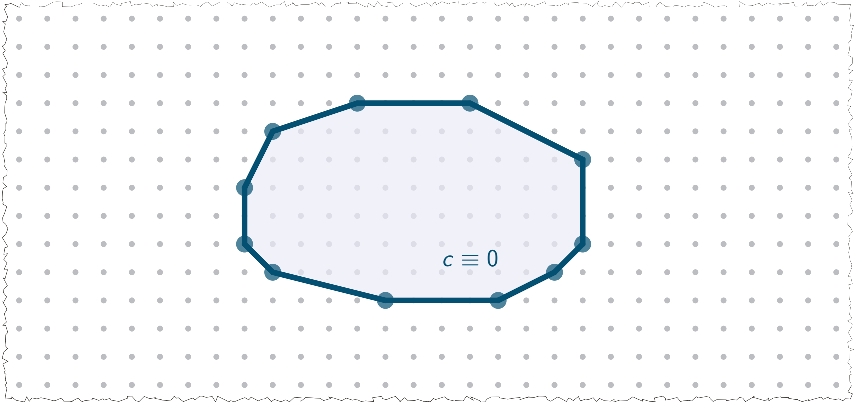



$c \equiv 0$

Can we say something about this region's shape ?



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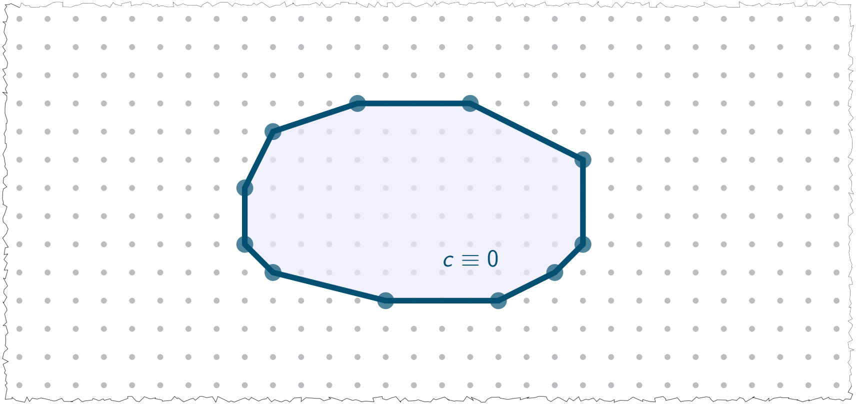

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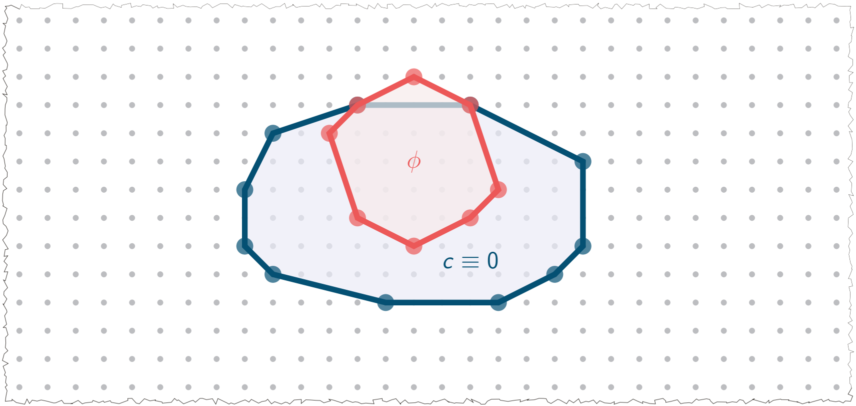

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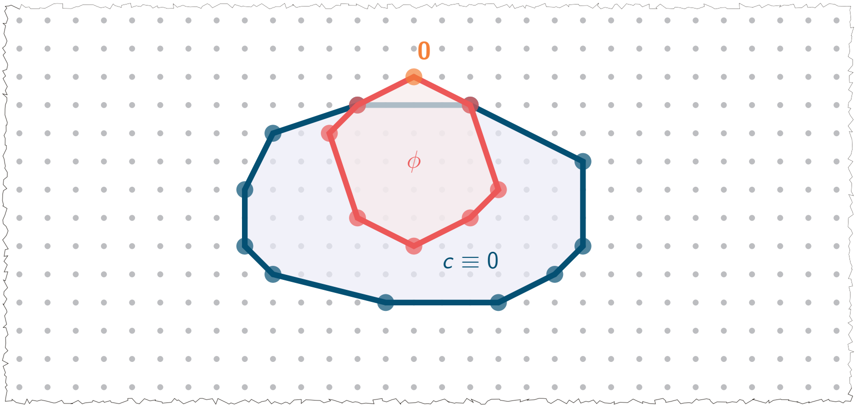

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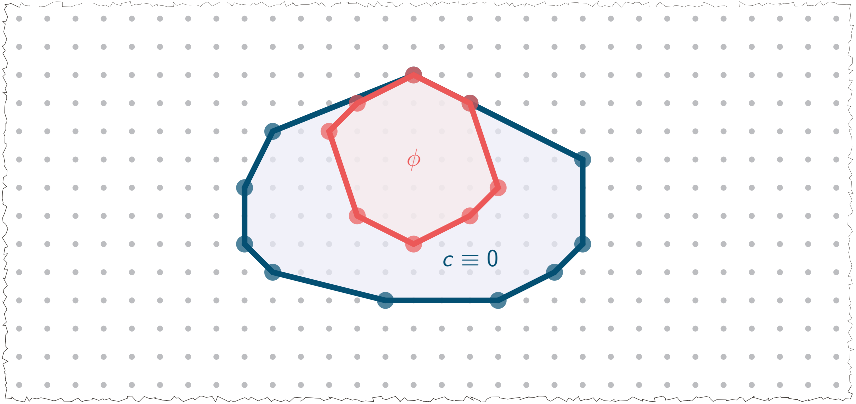

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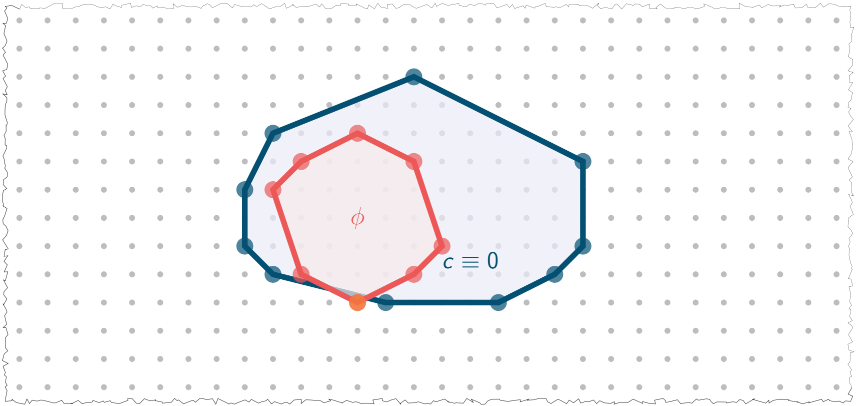

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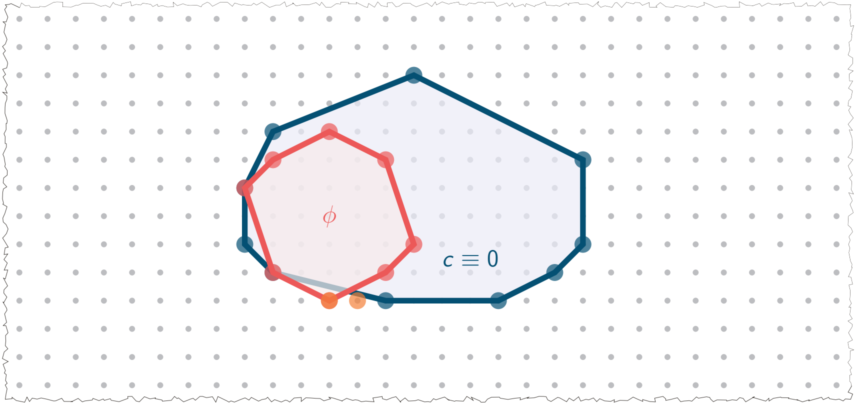

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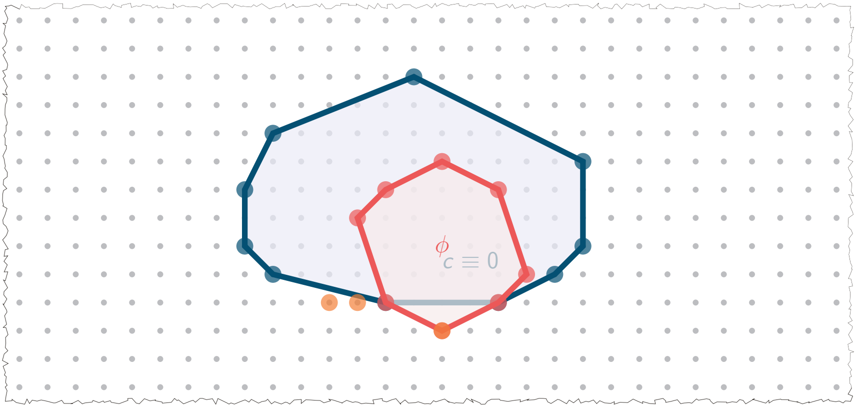

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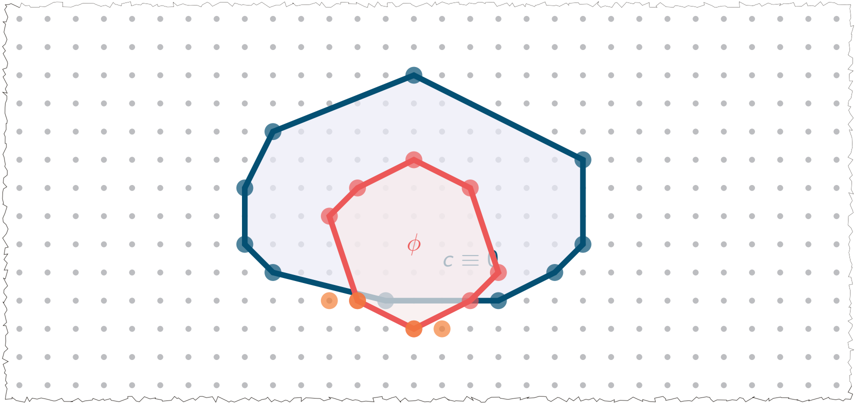

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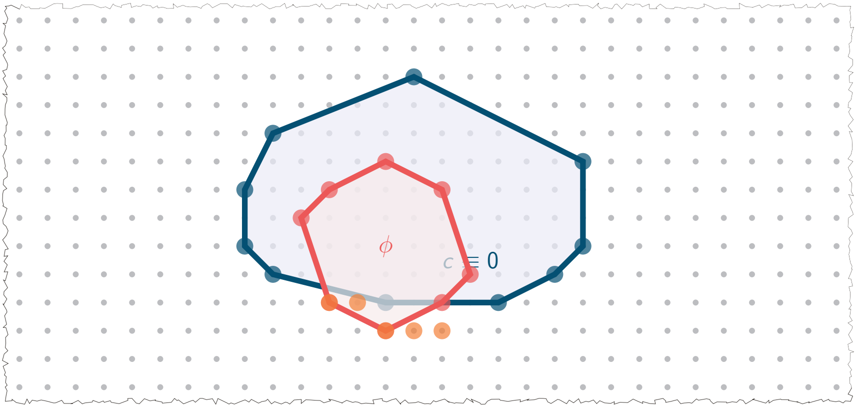

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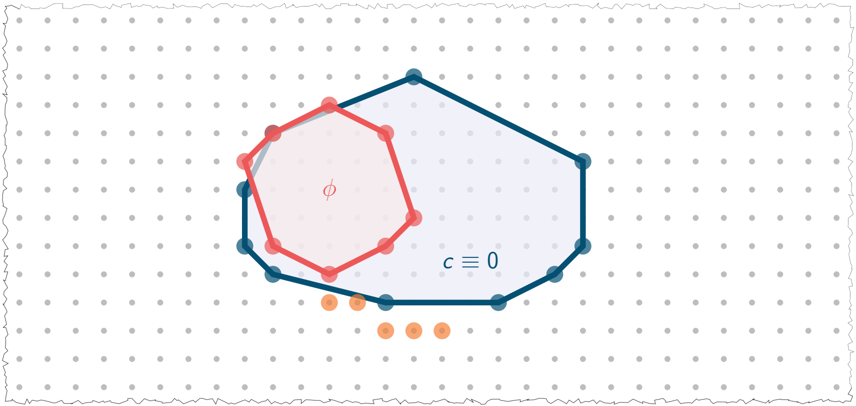

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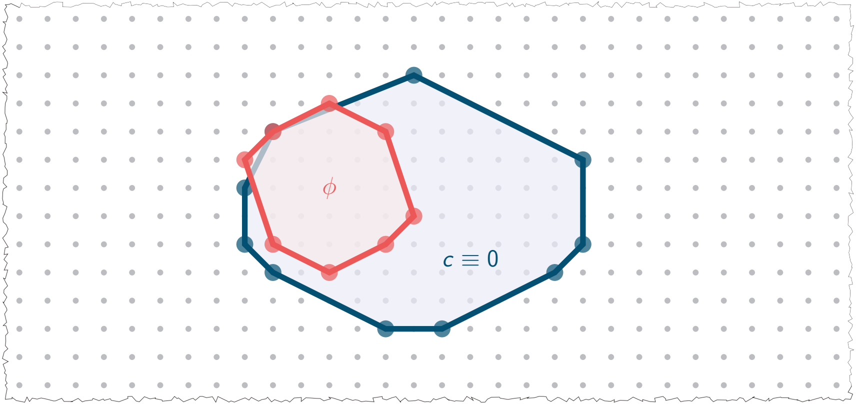

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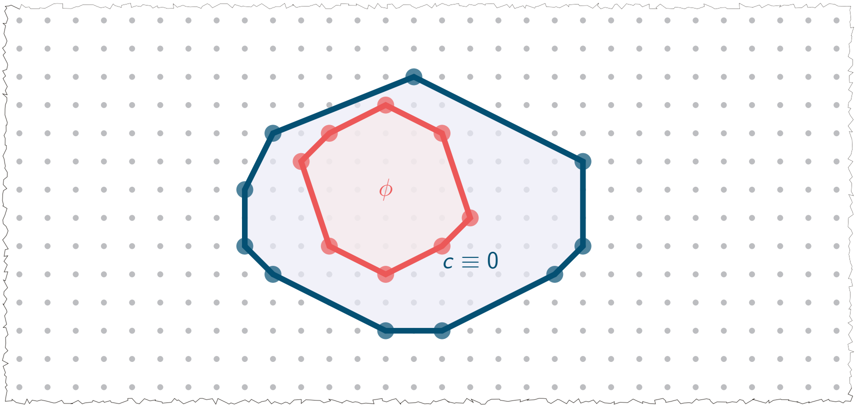

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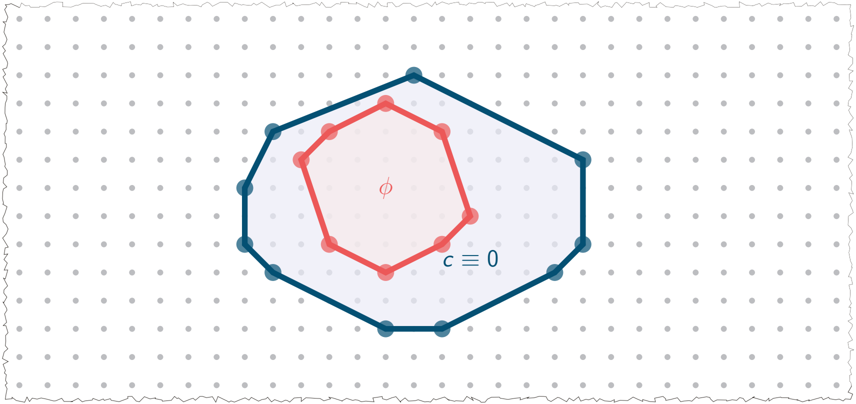

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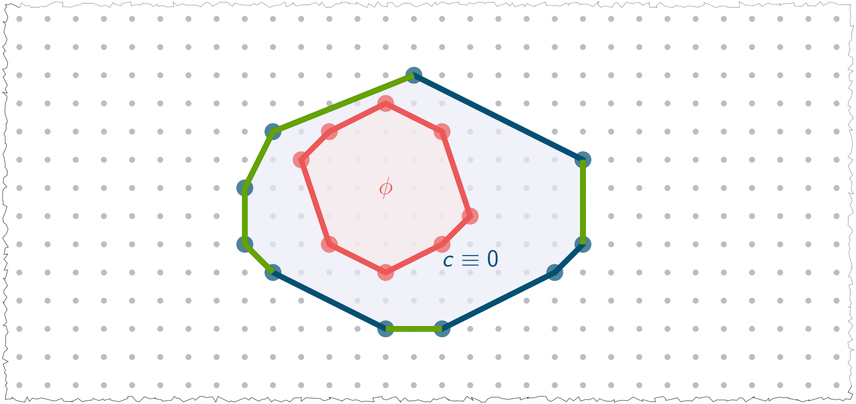

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How does a shape evolve under the expansion process ?

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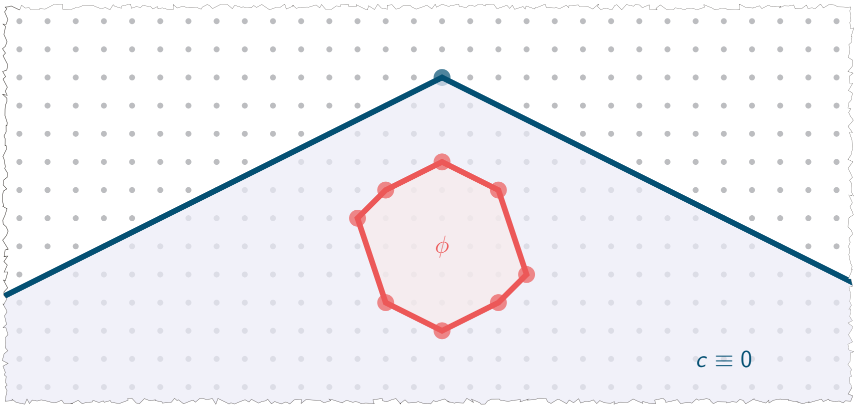

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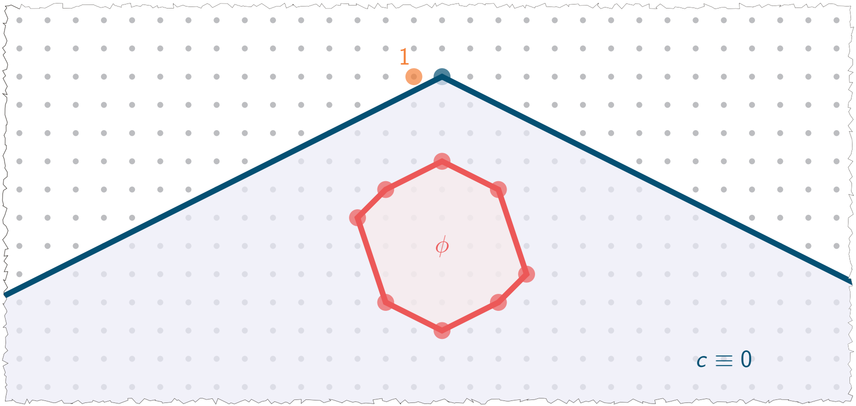

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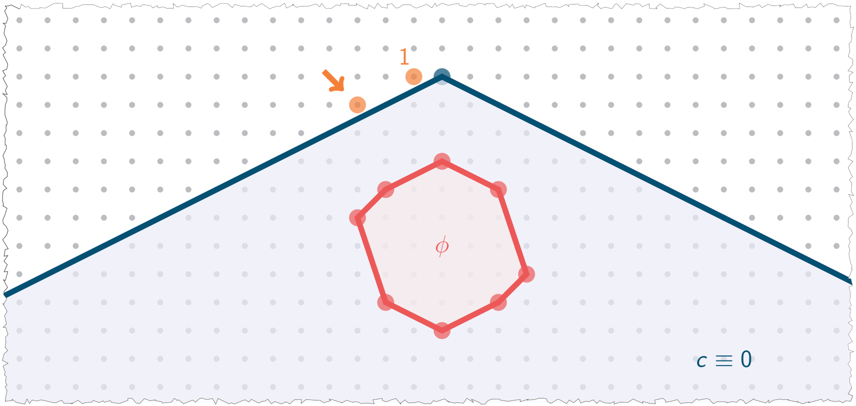

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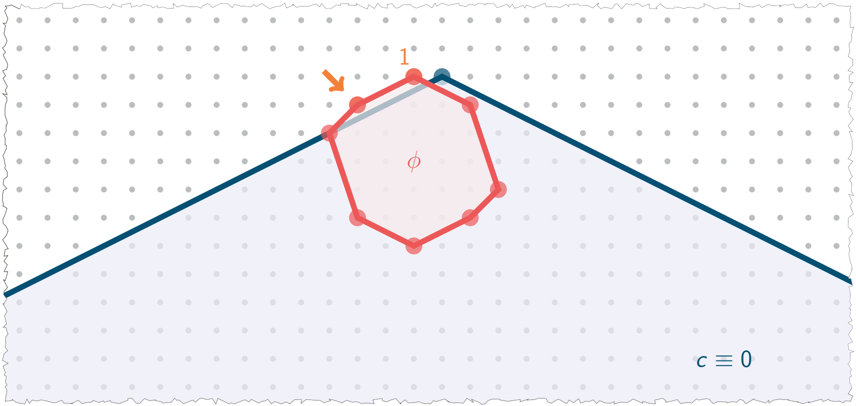

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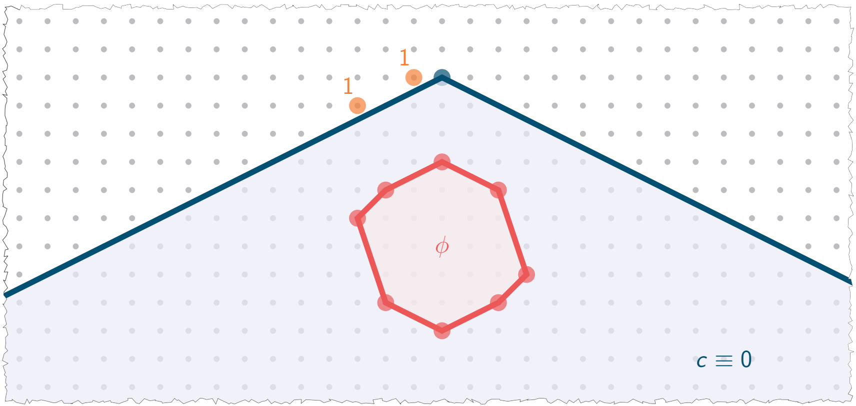

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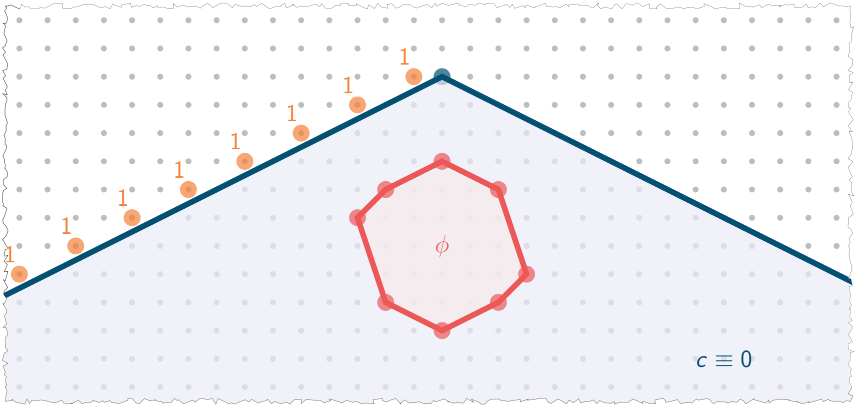

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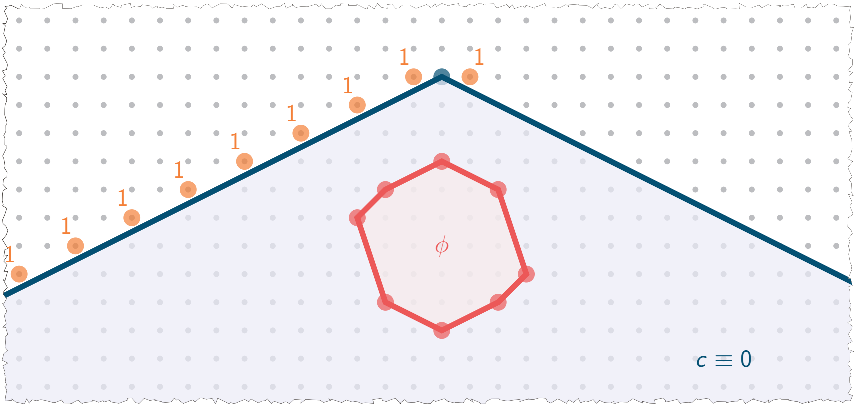

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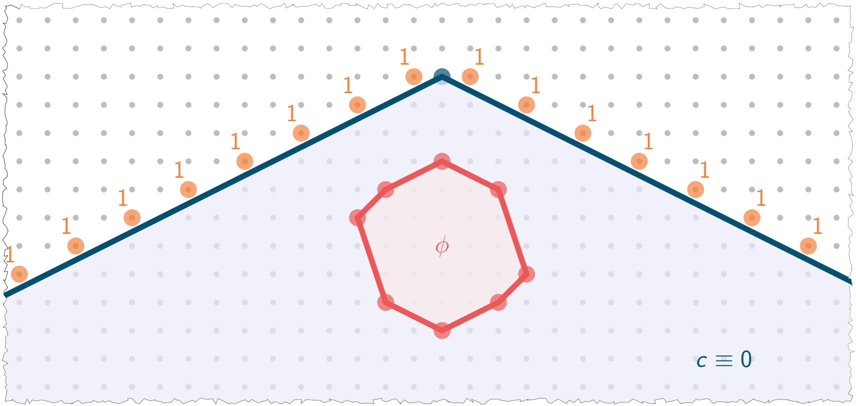

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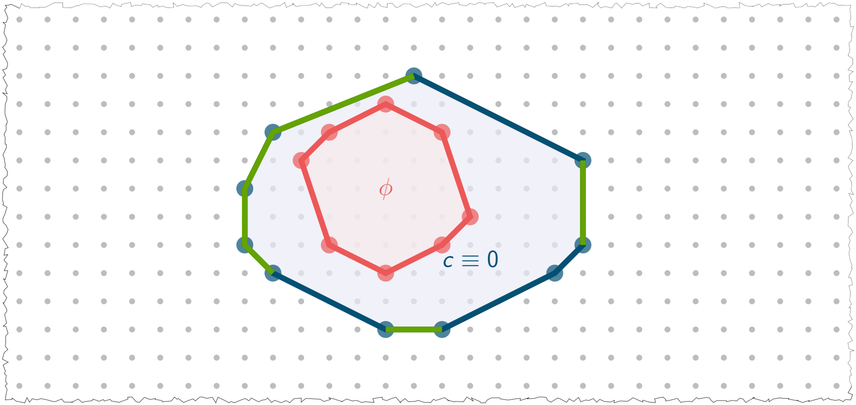

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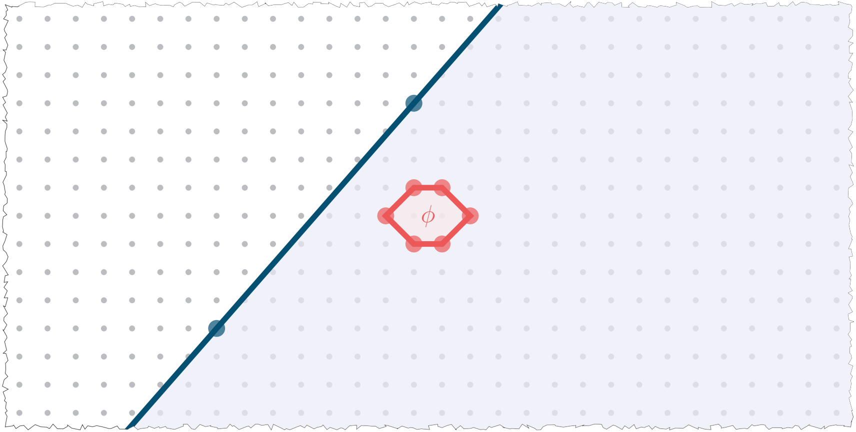

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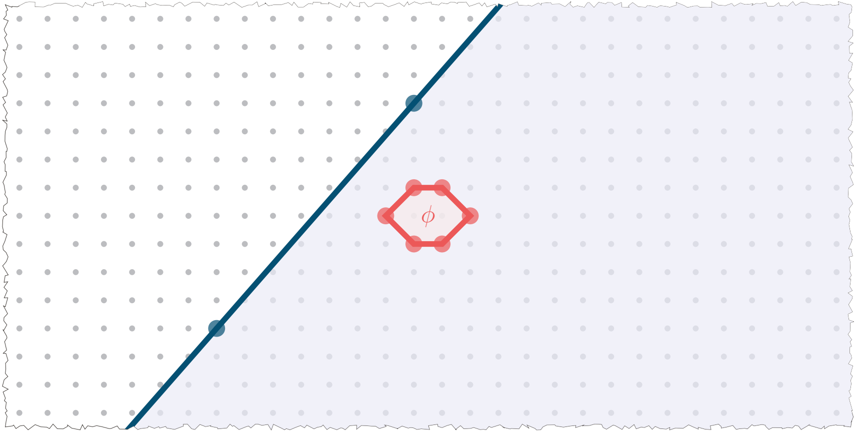

What's true: A non-parallel hyperplan cannot be stable.

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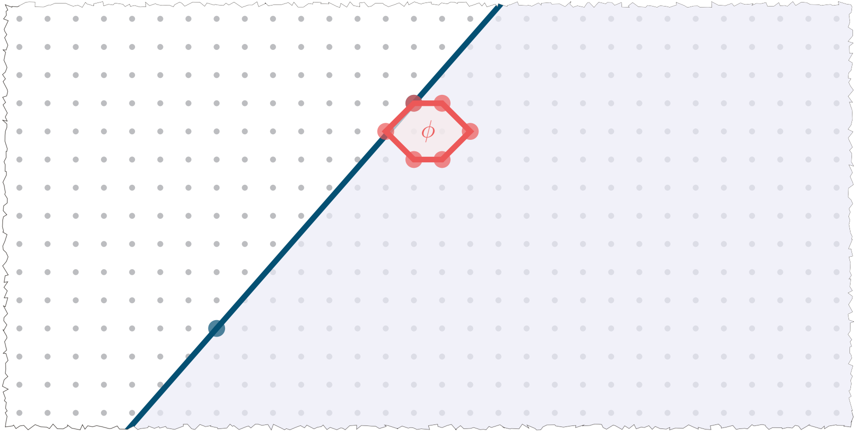

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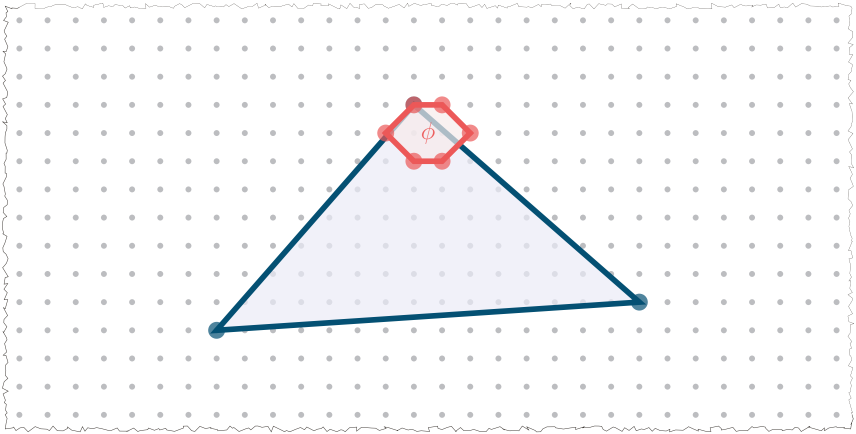

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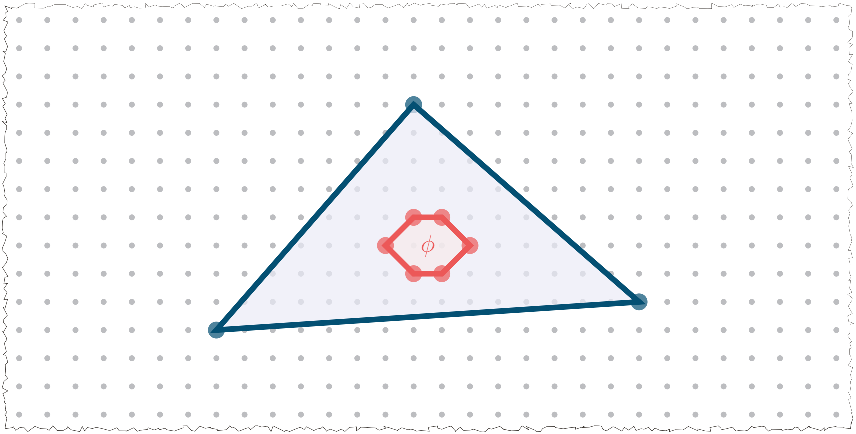

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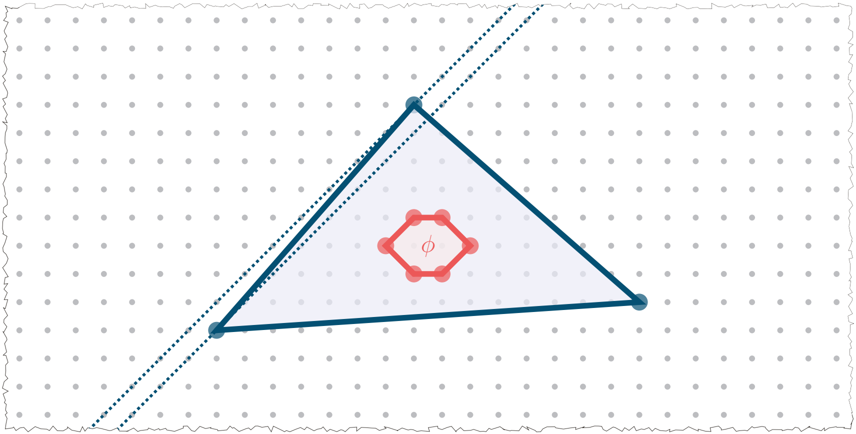

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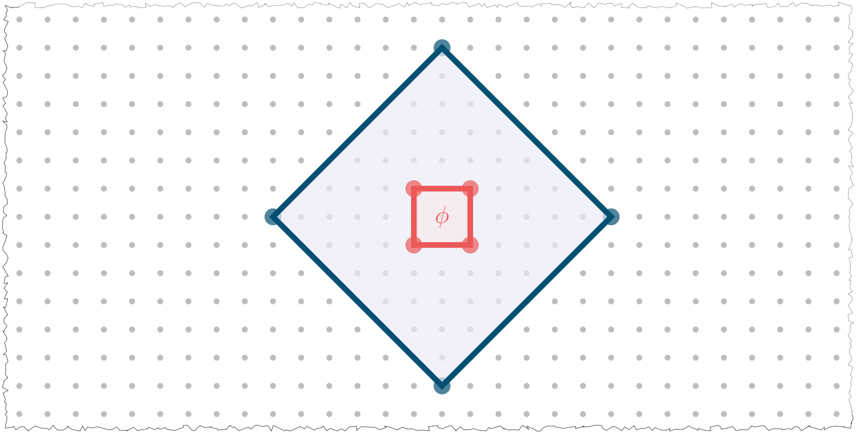

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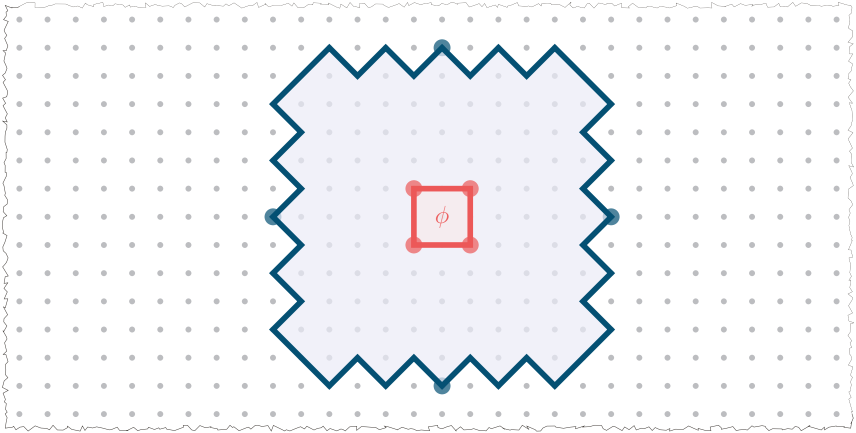

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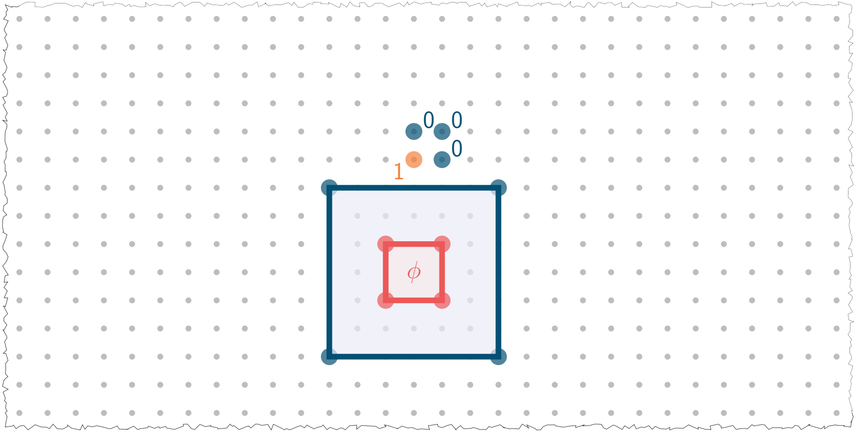

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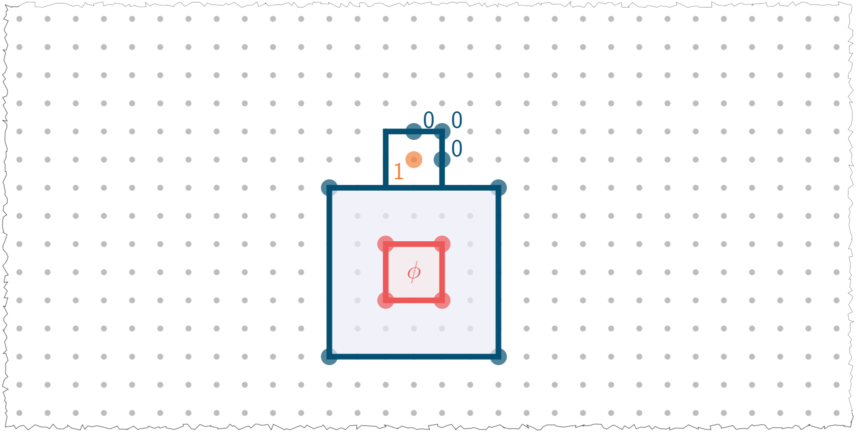

So, what do we do?

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We find a maximal stable shape using Zorn's lemma ¯_(ツ)_/¯

Dziękuję

References I

Jarkko Kari and Etienne Moutot, *Decidability and periodicity of low complexity tilings*, Theory of Computing Systems (2021).

Marston Morse and Gustav A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics **60** (1938), no. 4, 815–866.

Michal Szabados, *An algebraic approach to nivat's conjecture*, Thèse de doctorat, University of Turku, Août 2018.

Morse-Hedlund Theorem

Theorem (Morse,Hedlund [MH38])

A bi-infinite word $c \in \mathcal{A}^{\mathbb{Z}}$ is periodic iff there exists $n \in \mathbb{N}$ such that

$$P_c(n) \leqslant n.$$

3D counterexample