Membership problems in groups

Markus Lohrey

University of Siegen

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Rational sets in arbitrary monoids: Definition 2

A finite automaton over M is a tuple $A = (Q, \delta, q_0, F)$ where

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Then: $L \in \mathrm{Rat}(M) \iff \exists$ finite automaton A over M : $L(A) = L$.

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Decidability/complexity does not depend on the chosen generating set.

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The subgroup membership problem is a widely studied problem in combinatorial group theory.

Simple implications

Clearly, for a f.g. group G we have:

$$\begin{aligned} & \text{RatMP}(G) \text{ decidable} \\ \Rightarrow\quad & \text{SubmonoidMP}(G) \text{ decidable} \\ \Rightarrow\quad & \text{SubgroupMP}(G) \text{ decidable} \end{aligned}$$

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We will come back to this latter.

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The knapsack problem for G is the following computational problem:

INPUT: A finite list $g_1, \ldots, g_n, g$ of elements from G.

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Open problem: Is there a group, for which $\mathbb{N}$-knapsack is undecidable and $\mathbb{Z}$-knapsack is decidable?

Uniform versus non-uniform membership problems

Theorem (Bodart 2025)

There is a finitely generated group G with the following properties:

- SubgroupMP(G) is undecidable.
- For every fixed finitely generated subgroup $H \leq G$, it is decidable whether a given element of G belongs to H.
- For every fixed finitely generated submonoid $M \subseteq G$, it is decidable whether a given element of G belongs to M.

Bodart's group G is a central extension of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ by $\mathbb{Z}_2$: $G/\mathbb{Z}_2 \cong \mathbb{Z}_2 \wr \mathbb{Z}$ with $\mathbb{Z}_2$ central in G.

Open problem: Is there a group G with RatMP(G) undecidable but membership in every fixed rational subset of G is decidable?

Extreme cases: abelian groups

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- Decidability (in **NP**) follows by a reduction to integer linear programming (Grunschlag 1999).
- For a fixed f.g. abelian group G, membership in **P** follows from the fact that integer programming in a fixed dimension is in **P** (Lenstra 1983).

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Nielsen 1921, Stallings 1983, Avenhaus and Madlener 1984

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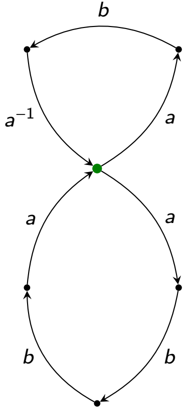

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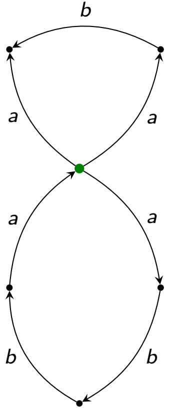

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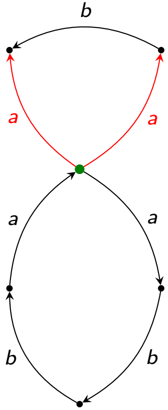

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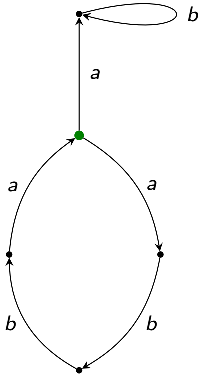

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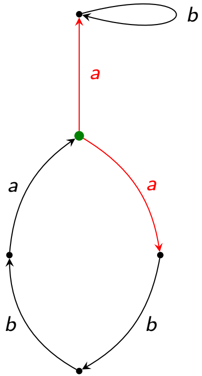

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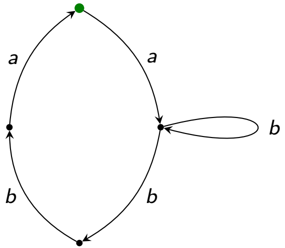

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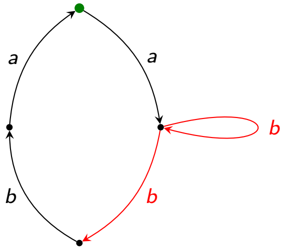

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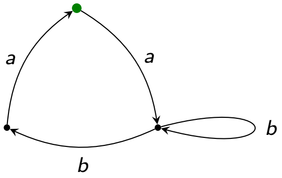

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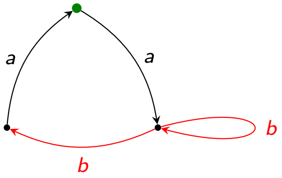

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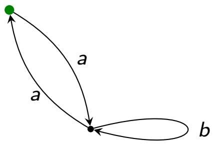
When the graph is completely folded, test membership of $abaaa$.



Compressed Stallings' folding

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Corollary (Lohrey 21)

SubgroupMP($\mathrm{GL}_2(\mathbb{Z})$) can be decided in polynomial time if all input matrices are given in binary notation.

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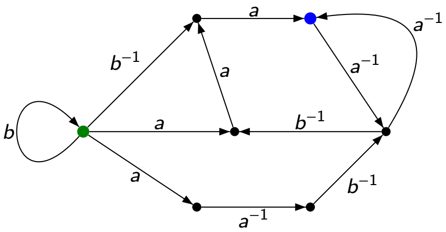

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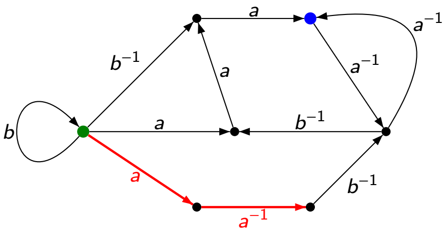

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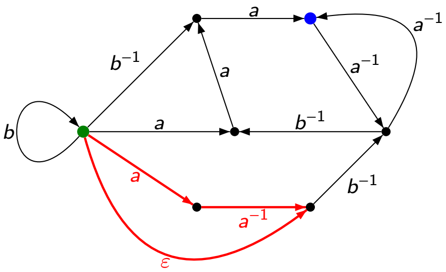

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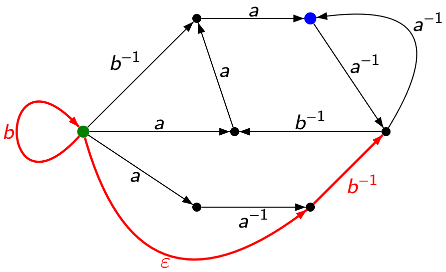

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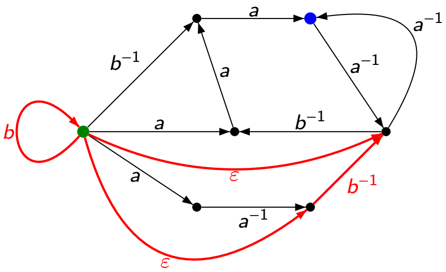

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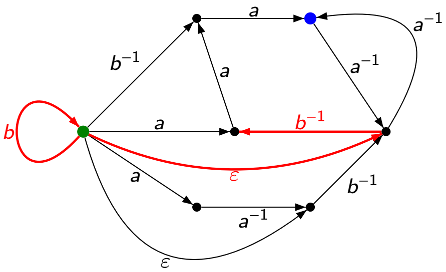

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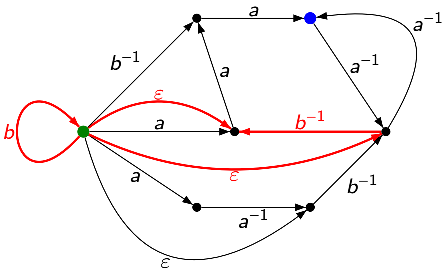

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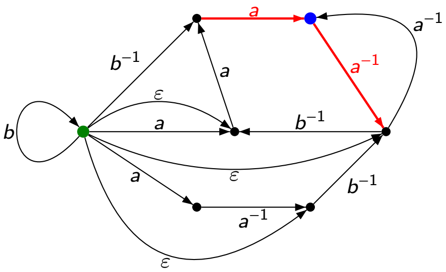

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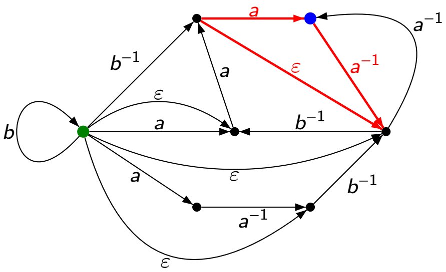

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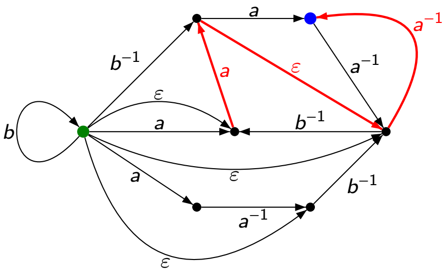

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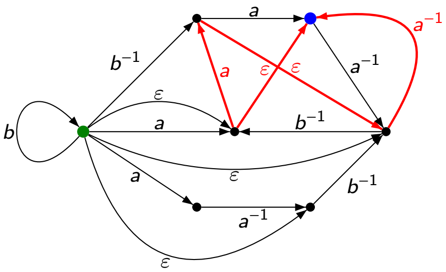

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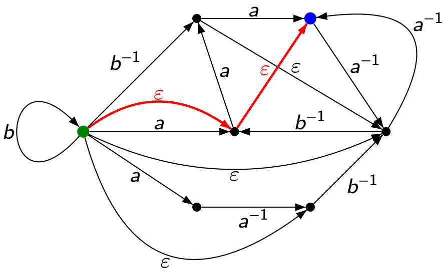

Interpolating between free and abelian: graph groups

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Every graph group is linear (a subgroup of $\mathrm{GL}_d(\mathbb{F})$ for a field $\mathbb{F}$).

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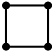
Two important graphs: P4:  C4: 

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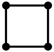
Two important graphs: P4:  C4: 

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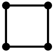
Two important graphs: P4:  C4: 

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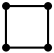
Two important graphs: P4:  C4: 

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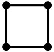
Two important graphs: P4:  C4: 

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Graph groups: SubmonoidMP and RatMP

L, Steinberg 2006

The following are equivalent for every finite undirected graph (A, E):

- RatMP$(G(A, E))$ is decidable.
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The graphs that neither contain an induced P4 or C4 are exactly the graphs that can be constructed from the single-node graphs • using

- disjoint union and
- adding a new vertex • and connecting it to all other vertices.

Graph groups: SubmonoidMP and RatMP

Consequence: The graph groups $G(A, E)$, where (A, E) neither contains an induced P4 nor an induced C4, form the smallest class C of groups with:

- $\mathbb{Z} \in \mathsf{C}$,
- if $G, H \in \mathsf{C}$ then $G * H \in \mathsf{C}$ ($G * H$ is the free product)
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Haase, Zetzsche 2019

Let (A, E) be a finite graph that neither contains an induced P4 nor an induced C4.

- RatMP($G(A, E)$) is **NP**-complete if (A, E) contains an induced P3.
- RatMP($G(A, E)$) is in **P** if (A, E) contains no induced P3.

Submonoids versus subgroups

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What about submonoids versus rational subsets?

Slightly non-abelian: nilpotent groups

A group G is nilpotent if the lower central series $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$ with $G_{i+1} = [G_i, G]$ finally reaches the trivial group 1.

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The problem is actually in $\mathbf{TC}^0$ (Myasnikov and Weiß 2022).

Heisenberg groups: decidability

The discrete n-dimensional Heisenberg group ($n \geq 3$) is

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Colcombet, Ouaknine, Semukhin, Worrell 2019

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Proof (for $n = 3$): Take matrixes $A, A_1, \ldots, A_k \in \mathsf{H}_3(\mathbb{Z})$ and let

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We get:

$$A_i^n = \begin{pmatrix} 1 & a_i \cdot n & c_i \cdot n + a_i b_i \frac{(n-1)n}{2} \\ 0 & 1 & b_i \cdot n \\ 0 & 0 & 1 \end{pmatrix}$$

Heisenberg groups

There is a solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of $A = A_1^{x_1} \cdots A_k^{x_k}$ iff the following system of three Diophantine equations has a solution over $\mathbb{Z}$:

$$a = \sum_{i=1}^{k} a_i \cdot x_i$$

$$b = \sum_{i=1}^{k} b_i \cdot x_i$$

$$c = \sum_{i=1}^{k} c_i \cdot x_i + \sum_{i=1}^{k} a_i b_i \frac{(x_i - 1)x_i}{2} + \sum_{1 \le i < j \le k} a_i b_j x_i x_j$$

This is a Diophantine system with a single quadratic equation and two linear equations.

Decidability follows by results from Duchin, Liang, and Shapiro 2015 (based on Grunewald and Segal 2004). □

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Bodart 2024

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Open problem: Does this generalize to every $H_n(\mathbb{Z})$ for $n \geq 3$?

Nilpotent groups: undecidability

König, L, Zetzsche 2014

There exist nilpotent groups of class 2 with undecidable knapsack problem.

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There is a fixed polynomial $P(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$ such that the following problem is undecidable:

- INPUT: $a \in \mathbb{N}$.
- QUESTION: $\exists (x_1, \ldots, x_k) \in \mathbb{Z}^k : P(x_1, \ldots, x_k) = a$?

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Write $P(X_1, \dots, X_k) = a$ as a system $\mathcal{S}$ of equations of the form

$$X \cdot Y = Z, \ X + Y = Z, \ X = c \ (c \in \mathbb{Z})$$

with a distinguished equation $X_0 = a$.

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For $A \in \mathsf{H}_3(\mathbb{Z})$ let $A_1 = (A, \mathsf{Id}, \mathsf{Id})$, $A_2 = (\mathsf{Id}, A, \mathsf{Id})$, $A_3 = (\mathsf{Id}, \mathsf{Id}, A)$.

Nilpotent groups: undecidability

The solutions of $\{X_0 = a,\ X_0 = X \cdot Y,\ Y = X + Z\}$ are the solutions of

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_1^a =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}_2^X \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_2^Y \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}_2^X \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_2^Y \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_2^{X_0} \cdot$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_3^X \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_3^Z \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_3^Y$$

Nilpotent groups: undecidability

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Example: Consider an equation

$$g = a^Y b^Z c^Y d^Z$$

with $g, a, b, c, d \in G$ (any group).

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It has a solution (with $Y, Z \in \mathbb{Z}$ if and only if the following equation (over the group $G \times \mathbb{Z}^4$) has a solution:

$$\begin{aligned}&(g,0,0,0,0) = \\ &\quad (\mathbf{1},1,0,1,0)^Y (\mathbf{1},0,1,0,1)^Z \\ &\quad (a,-1,0,0,0)^U (b,0,-1,0,0)^V (c,0,0,-1,0)^W (d,0,0,0,-1)^X\end{aligned}$$

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In our example: Work in $\mathsf{H}_3(\mathbb{Z})^3 \times \mathbb{Z}^9 \leq \mathsf{H}_3(\mathbb{Z})^{12}$ (still nilpotent of class 2).

Nilpotent groups: undecidability

What we actually proved:

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König, L 2014

There is a fixed class-2 nilpotent group G and a fixed sequence of elements $g_1, g_2, \ldots, g_n \in G$ such that membership in the product

$$\langle g_1 \rangle \langle g_2 \rangle \cdots \langle g_n \rangle$$

is undecidable.

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is undecidable.

Most of the g_i are central, which allows to write $\langle g_1 \rangle \langle g_2 \rangle \cdots \langle g_n \rangle$ as a product $G_1 G_2 G_3 G_4$ of four abelian subgroups of G.

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There is a class-2 nilpotent group G with four abelian subgroups G_1, G_2, G_3, G_4 such that membership in $G_1 G_2 G_3 G_4$ is undecidable.

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There is a class-2 nilpotent group G with three abelian subgroups G_1, G_2, G_3 such that membership in $G_1 G_2 G_3$ is undecidable.

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On the other hand:

Lennox, Wilson 1979

For every f.g. nilpotent (in fact, every polycyclic) group G the following problem is decidable:

INPUT: Two f.g. subgroups $G_1, G_2 \leq G$ and $g \in G$

QUESTION: $g \in G_1 G_2$?

Nilpotent groups: undecidability

Romankov 2023

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Romankov's proof is also based on a reduction from Hilbert 10 (quite technical).

Nilpotent groups: undecidability

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Romankov's proof is also based on a reduction from Hilbert 10 (quite technical).

Lefaucheux (unpublished work) gave a relatively direct reduction from knapsack in $\mathsf{H}_3(\mathbb{Z})^d$ to SubmonoidMP($\mathsf{H}_3(\mathbb{Z})^e$) for some $e > d$.

Nilpotent groups: undecidability

Submonoids versus rational subsets

Bodart 2024

There is a f.g. nilpotent group of class 2 with the following properties:

- RatMP(G) is undecidable.
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Another group with decidable SubmonoidMP and undecidable RatMP was found bei Shafrir:

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Shafrir 2018

Every wreath product $G = A \wr (\mathbb{Z} \times \mathbb{Z})$ with A finite and abelian has the following properties:

- $\mathrm{RatMP}(G)$ is undecidable.
- $\mathrm{SubmonoidMP}(G)$ is decidable.

Groups, where SubmonoidMP and RatMP are equivalent

For a f.g. group G (finitely generated by $A \subseteq G$) define the Cayley-graph $C(G, A)$ as the graph

$$C(G, A) = (G, \{(g, ga) \mid a \in A \cup A^{-1}\}).$$

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The number of ends of G is the supremum of the set

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It belongs to $\mathbb{N} \cup \{\infty\}$.

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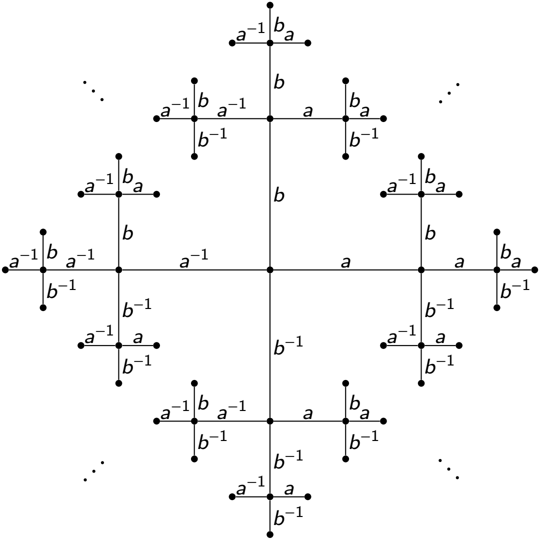
The number of ends is independent of the finite generating set A.

The ends of groups

A group with infinitely many ends: $F(a, b)$ (free group of rank 2)

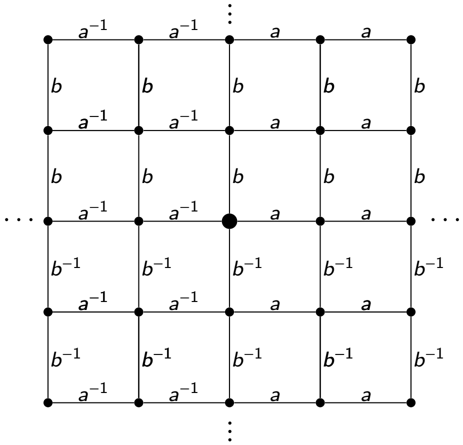


The ends of groups

A group with two ends: $\mathbb{Z}$

The ends of groups

A group with one end: $\mathbb{Z} \times \mathbb{Z}$



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- The proof uses the theory of groups acting on trees.

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Remarks:

- The proof uses the theory of groups acting on trees.
- If G has exactly two ends, then $\mathbb{Z}$ is a finite index subgroup of G.
  It is then easy to show that $\mathrm{RatMP}(G)$ is decidable.

More groups with decidable RatMP

The Baumslag-Solitar group $\mathsf{BS}_{1,q}$ for $q \geq 2$ is

$$\mathsf{BS}_{1,q} = \langle a, t \mid t^{-1}at = a^q \rangle = \left\{ \begin{pmatrix} q^z & t \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z},\ t \in \mathbb{Z}[1/q] \right\}$$

Cadilhac, Chistikov, Zetzsche 2020

RatMP($\mathsf{BS}_{1,q}$) is decidable for every $q \geq 2$.

Remember that SubmonoidMP is undecidable for

$$\mathbb{Z} \wr \mathbb{Z} \cong \left\{ \begin{pmatrix} x^z & t \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z},\ t \in \mathbb{Z}[x, x^{-1}] \right\}.$$

L, Steinberg, Zetzsche 2015

RatMP($G \wr H$) is decidable for every finite group G and free group H.

Open problem

The context-free subset membership problem for a group G (CFMP(G)) can be defined analogously to the rational subset membership problem.

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Easy observations:

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What else can we say?

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What else can we say?

In particular: Is CFMP($H_3(\mathbb{Z})$) decidable?

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Why is $\mathsf{CFMP}(\mathsf{H}_3(\mathbb{Z}))$ is of particular interest?

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If G contains a free monoid $\{a, b\}^*$ then $\mathsf{CFMP}(G)$ is undecidable.

So, let G be virtually nilpotent.

If G is virtually abelian then $\mathsf{CFMP}(G)$ is decidable.

If G is not virtually abelian then G contains a copy of $\mathsf{H}_3(\mathbb{Z})$.