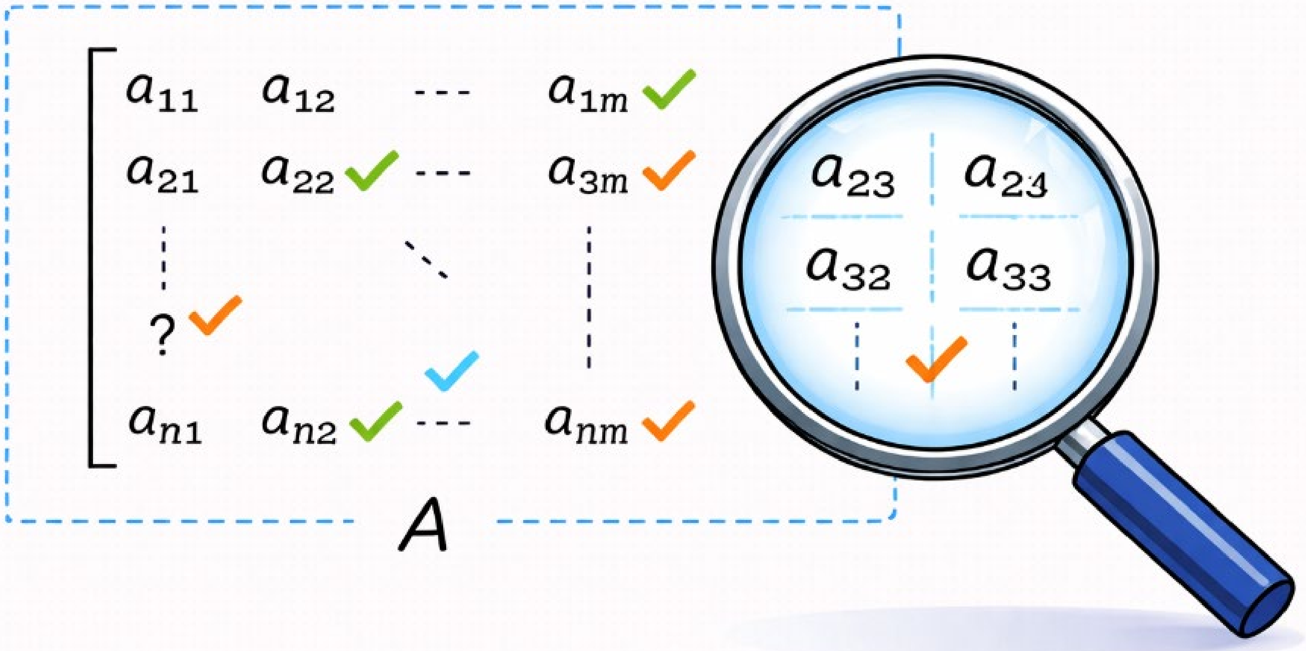
Mathematical and Computational Perspectives on the Binary and Boolean Rank and their Relation to the Real Rank

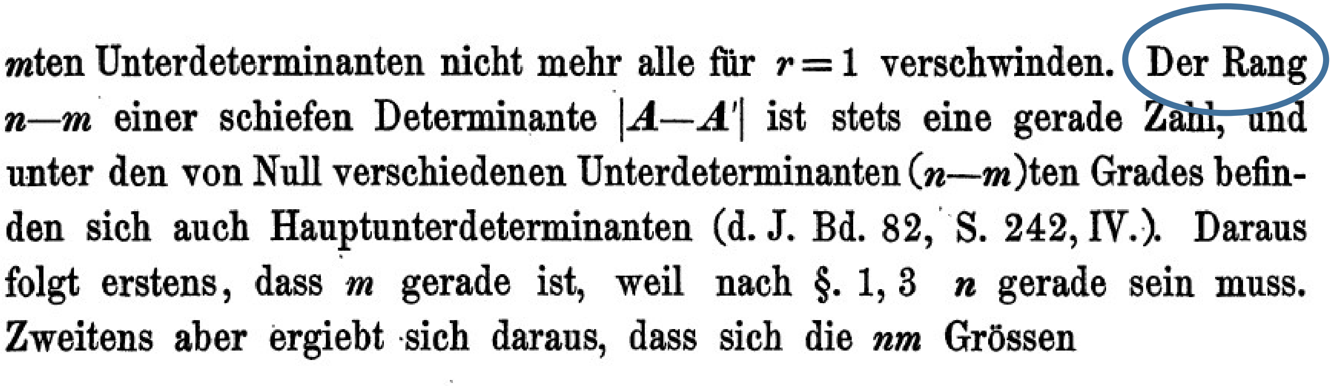


Michal Parnas

SAMSA 2026

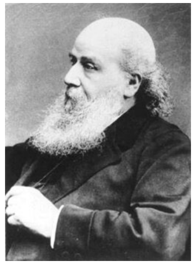
The Real Rank

- Originated from linear equations.
- Linear equations studied by Egyptians ~ 1600 B.C.
- 19’th century: modern study of matrices.
- Concept of rank defined.



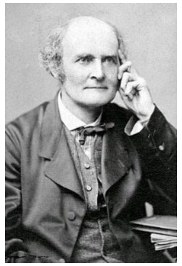
$m$ten Unterdeterminanten nicht mehr alle für $r=1$ verschwinden. Der Rang $n-m$ einer schiefen Determinante $|A-A'|$ ist stets eine gerade Zahl, und unter den von Null verschiedenen Unterdeterminanten $(n-m)$ten Grades befinden sich auch Hauptunterdeterminanten (d. J. Bd. 82, S. 242, IV.). Daraus folgt erstens, dass m gerade ist, weil nach §. 1, 3 n gerade sein muss. Zweitens aber ergiebt sich daraus, dass sich die nm Grössen

Frobenius, 1879: Rank is the largest **r** such that ∃ sub-matrix of size **rxr** with determinant $\neq 0$



Sylvester

Cauchy

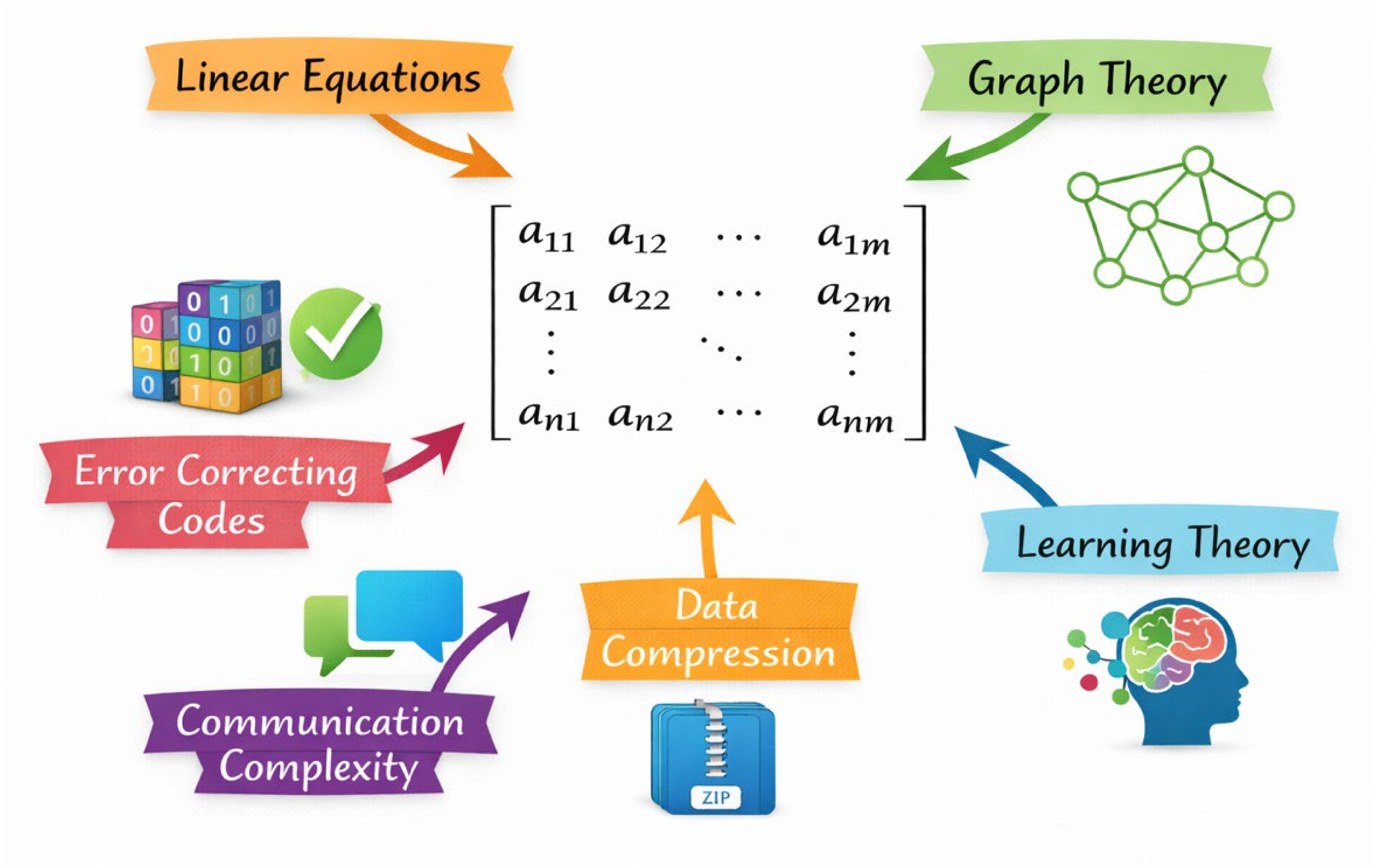


Cayley

Frobenius

Hamilton

Applications of Rank



Mathematics +

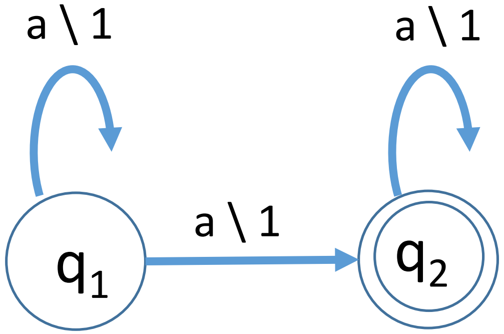
Computer Science

Automaton Example

Fliess, 1974: $f : \Sigma^* \to \mathbb{R}$ can be computed by a weighted finite automaton if and only if the Hankel matrix of f has finite rank n, where n = minimal number of states required.

$$\begin{array}{cc} & \begin{array}{cccccc} \varepsilon & a & aa & aaa & aaaa & \end{array} \\ \begin{array}{c} \varepsilon \\ a \\ aa \\ aaa \\ aaaa \\ \end{array} & \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & 3 & 4 & 5 & 6 & \cdots \\ 3 & 4 & 5 & 6 & 7 & \cdots \\ 4 & 5 & 6 & 7 & 8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

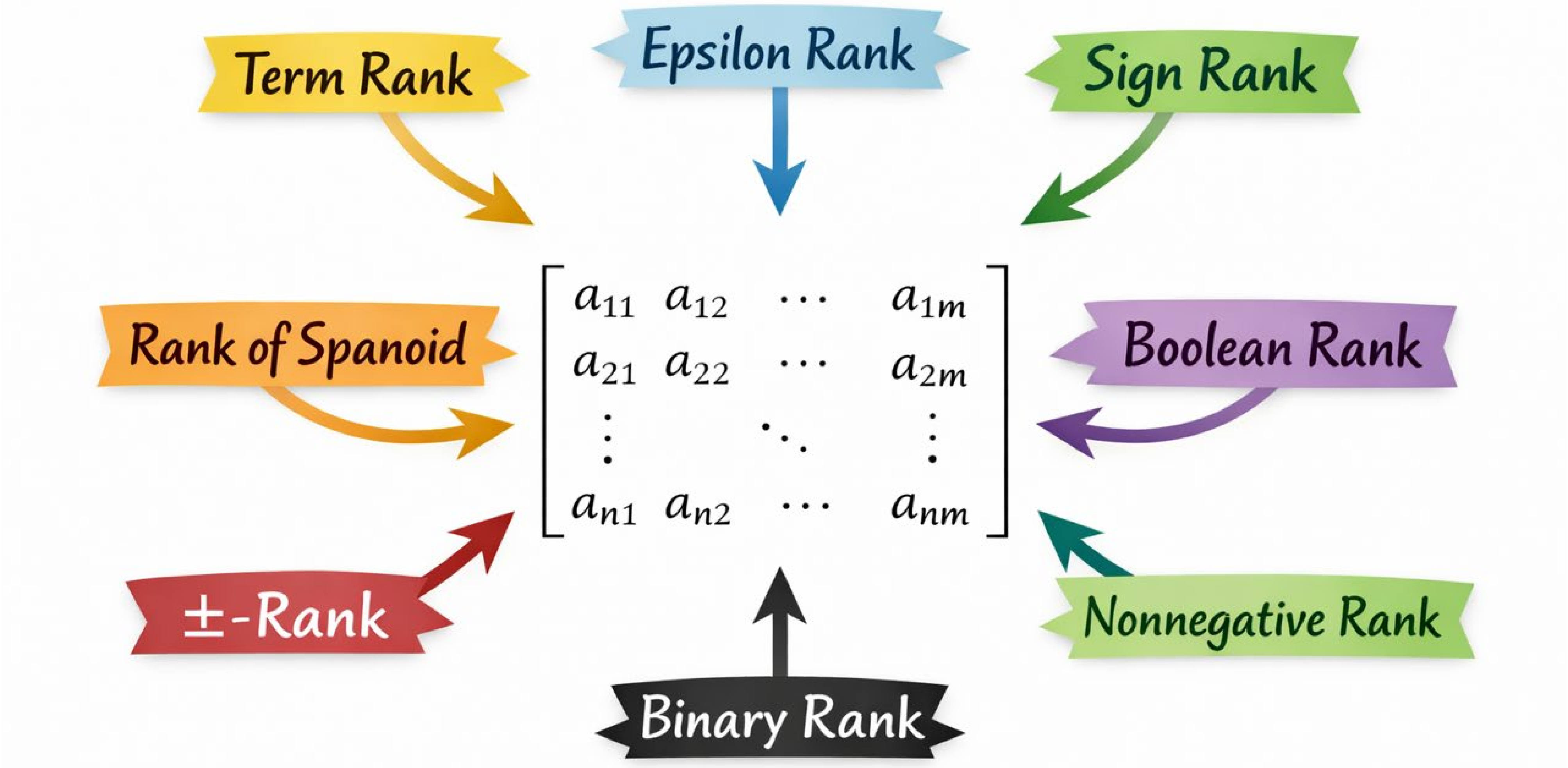
$f(a^n) = n$



$H_{u,v} = f(u \cdot v)$

$\mathrm{Rank}(H) = 2$

Rank Variations



$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The Real Rank: Alternative Definition

The real rank, Real($M_{n\times m}$), of a matrix $M_{n\times m}$ is the minimal d

such that $M_{n\times m}$ can be decomposed as: $M_{n\times m} = X_{n\times d} \cdot Y_{d\times m}$

$$\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{pmatrix}$$

The Binary and Boolean Rank

Binary/Boolean Rank of $M_{n\times m}$ is the minimal d such that $M_{n\times m} = X_{n\times d} \times Y_{d\times m}$,

where $M_{n\times m}, X_{n\times d}, Y_{d\times m}$ are **0,1** matrices and:

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- Binary($M_{n\times m}$): operations are over the reals (1 + 1 = 2).

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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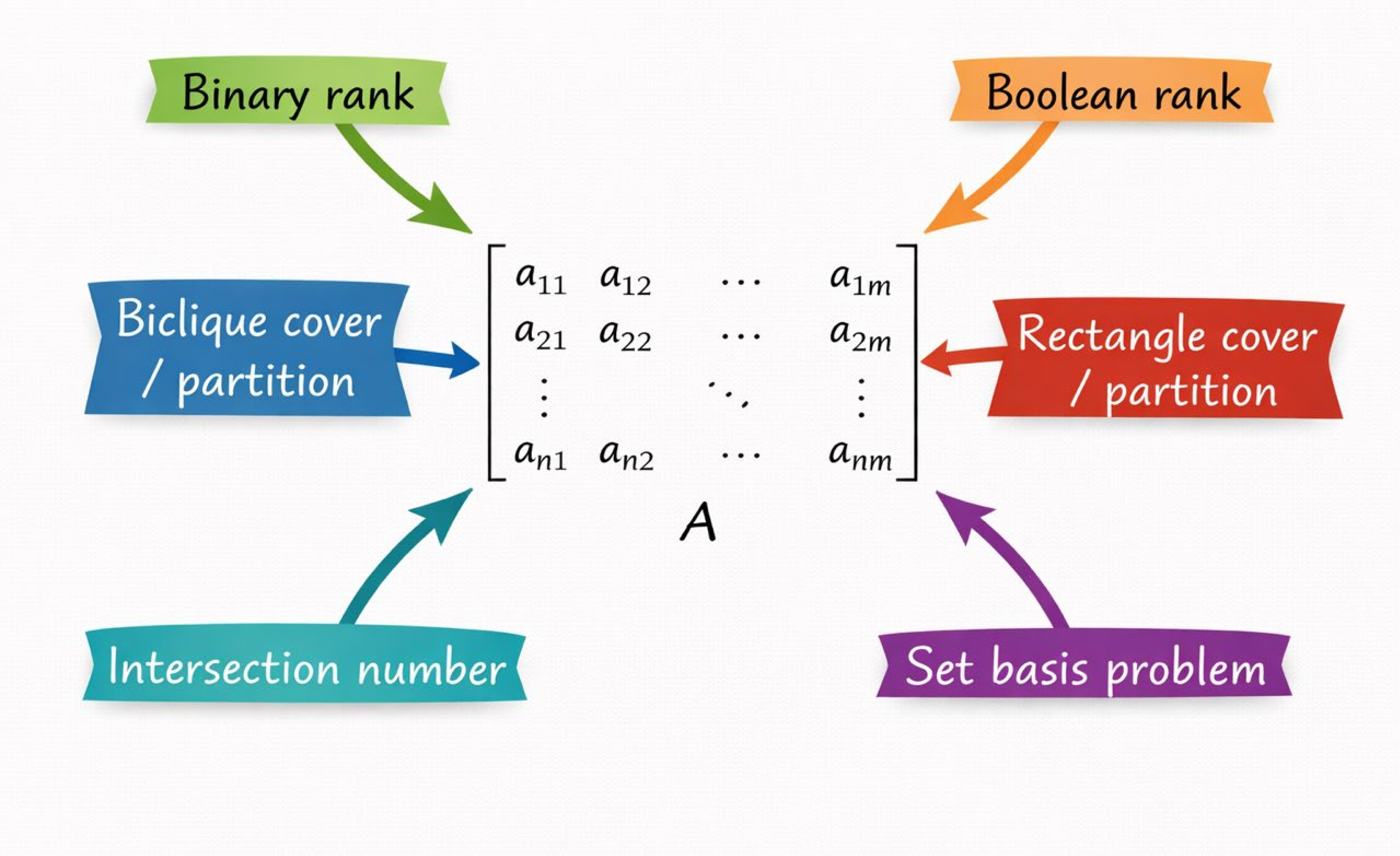
- Boolean($M_{n\times m}$): operations are Boolean (1 + 1 = 1).

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Equivalent Formulations

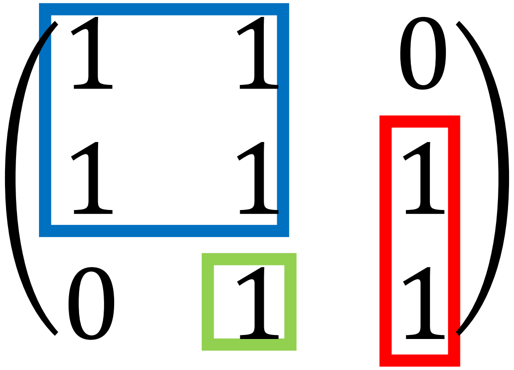


$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

A

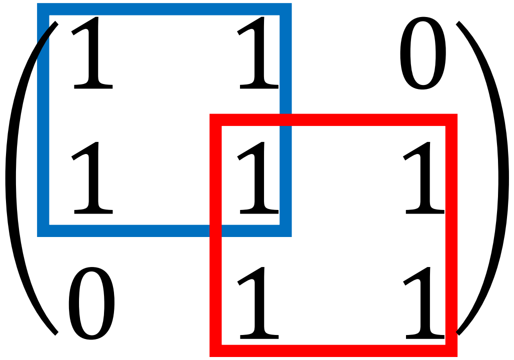
Rectangle Partition/Cover

Binary/Boolean rank = Minimal # monochromatic rectangles to partition/cover all 1’s of M.



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Binary(M) = 3



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Boolean(M) = 2

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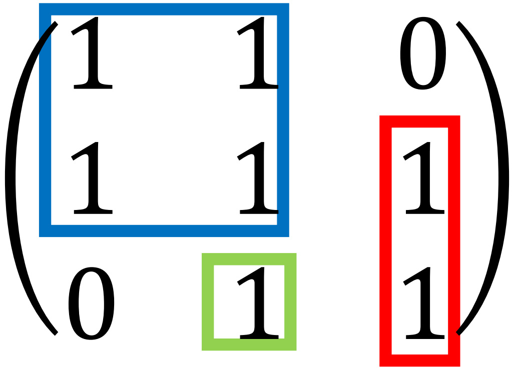

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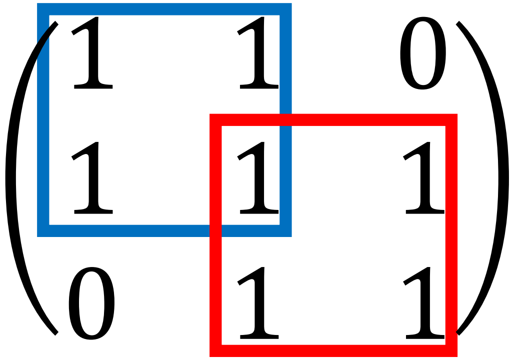


$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Boolean(M) = 2

Each rectangle is a product of a column and row vector: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$

Determines columns and rows of decomposition of matrix.

Biclique Partition/Cover

Binary/Boolean rank = Minimal # bicliques to partition/cover all edges of bipartite graph G = (L,R,E) represented by reduced adjacency matrix M.

$$M_{i,j} = \begin{cases} 1, & (i,j) \in E \\ 0, & else \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Binary(M) = 3

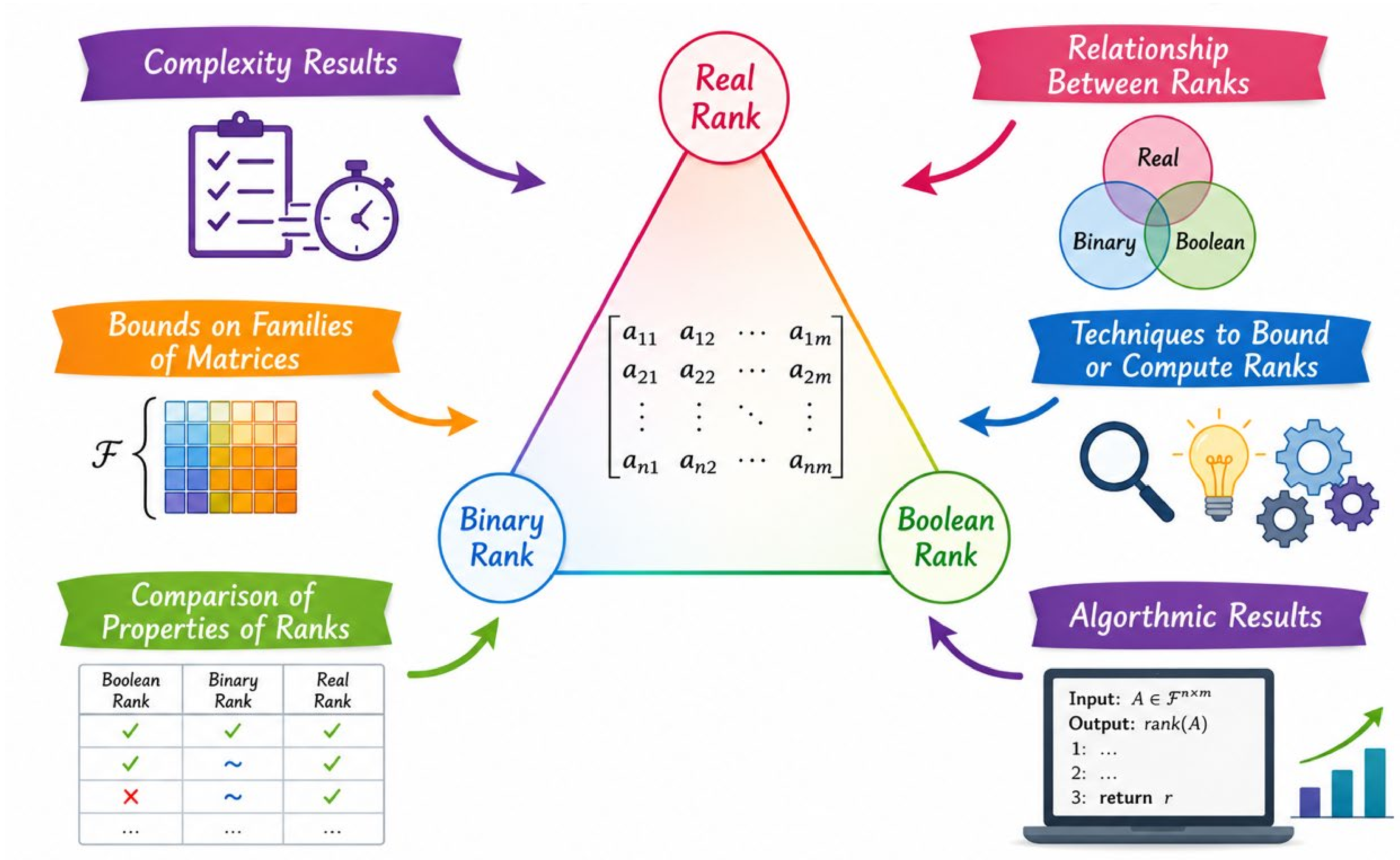
Partition(G) = 3

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Boolean(M) = 2

Cover(G) = 2

Research Directions



Complexity Results

Bounds on Families of Matrices

Comparison of Properties of Ranks

Boolean Rank	Binary Rank	Real Rank
✓	✓	✓
✓	~	✓
✗	~	✓
…	…	…

Real Rank

Binary Rank

Boolean Rank

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

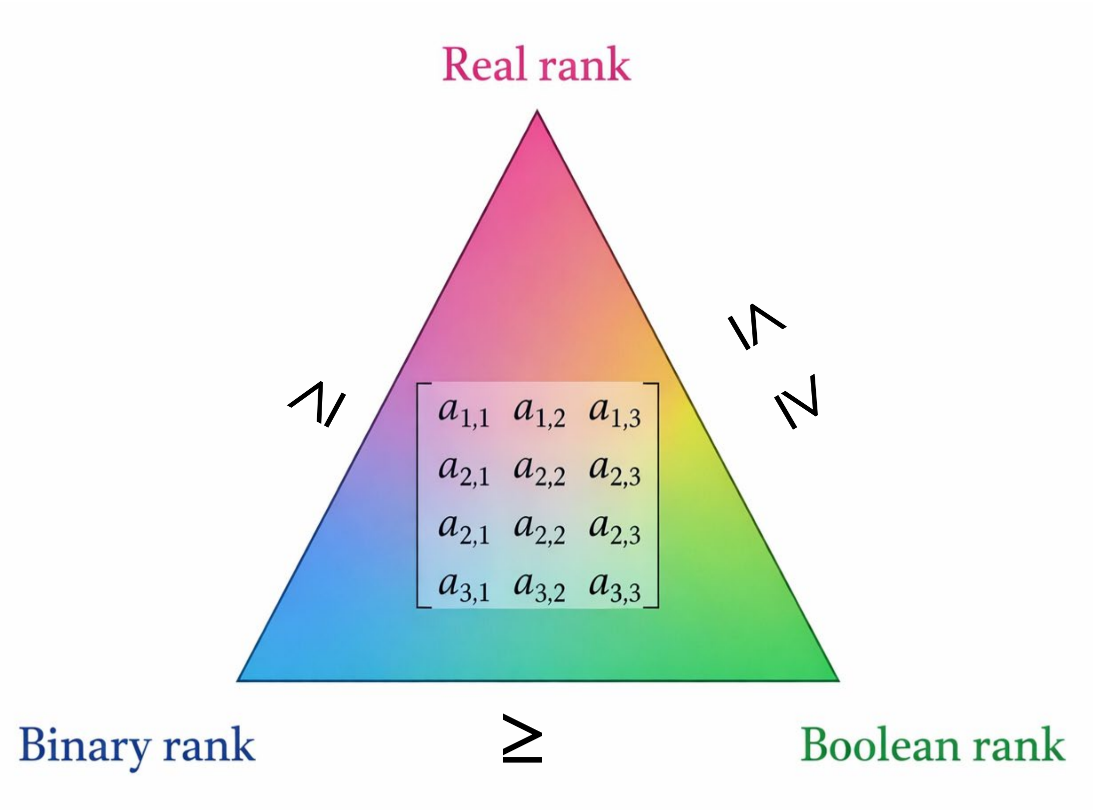
Relationship Between Ranks

Techniques to Bound or Compute Ranks

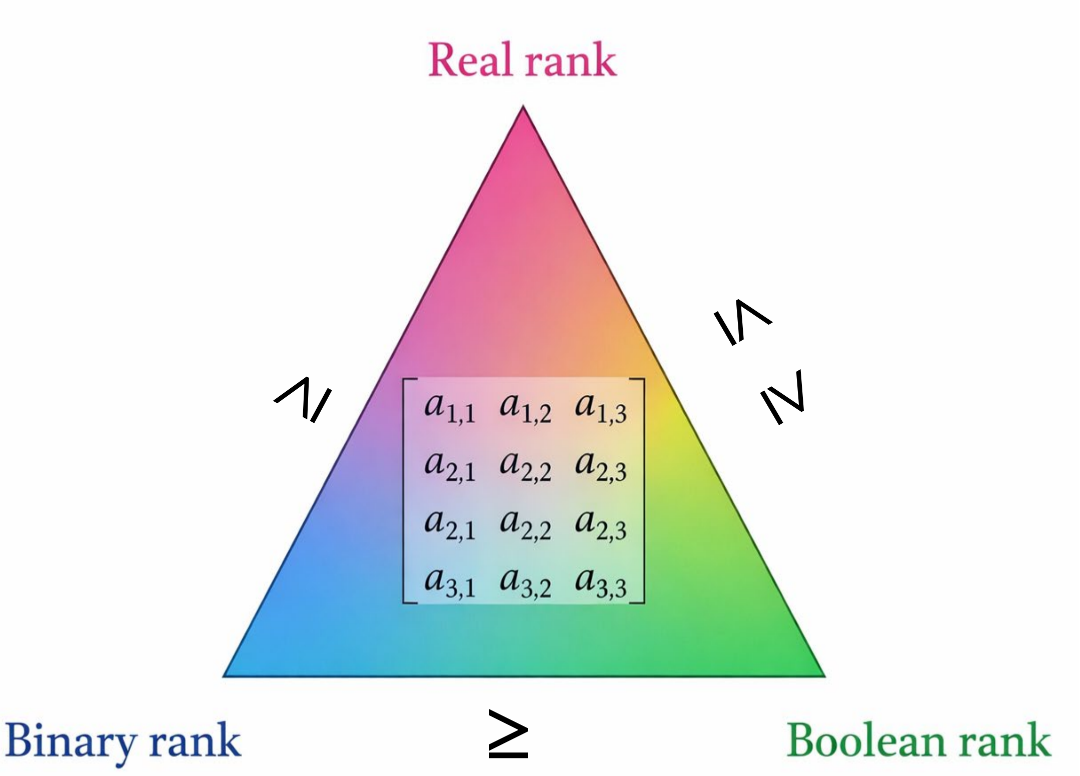
Algorthmic Results

```
Input: $A \in \mathcal{F}^{n \times m}$
Output: rank($A$)
…
…
return $r$
```

Relationships between Rank Functions



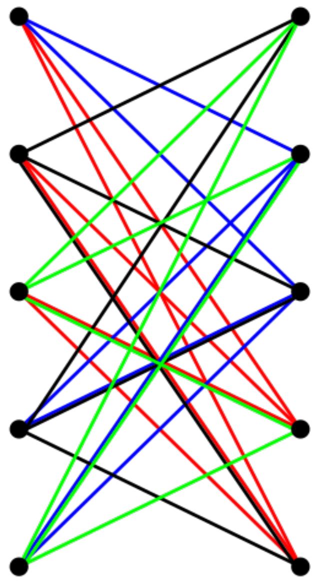
Relationships between Rank Functions



$$C_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Boolean(C_n) = $\Theta(\log_2 n)$

Real(C_n) = n, Binary(C_n) = n



Crown graph.

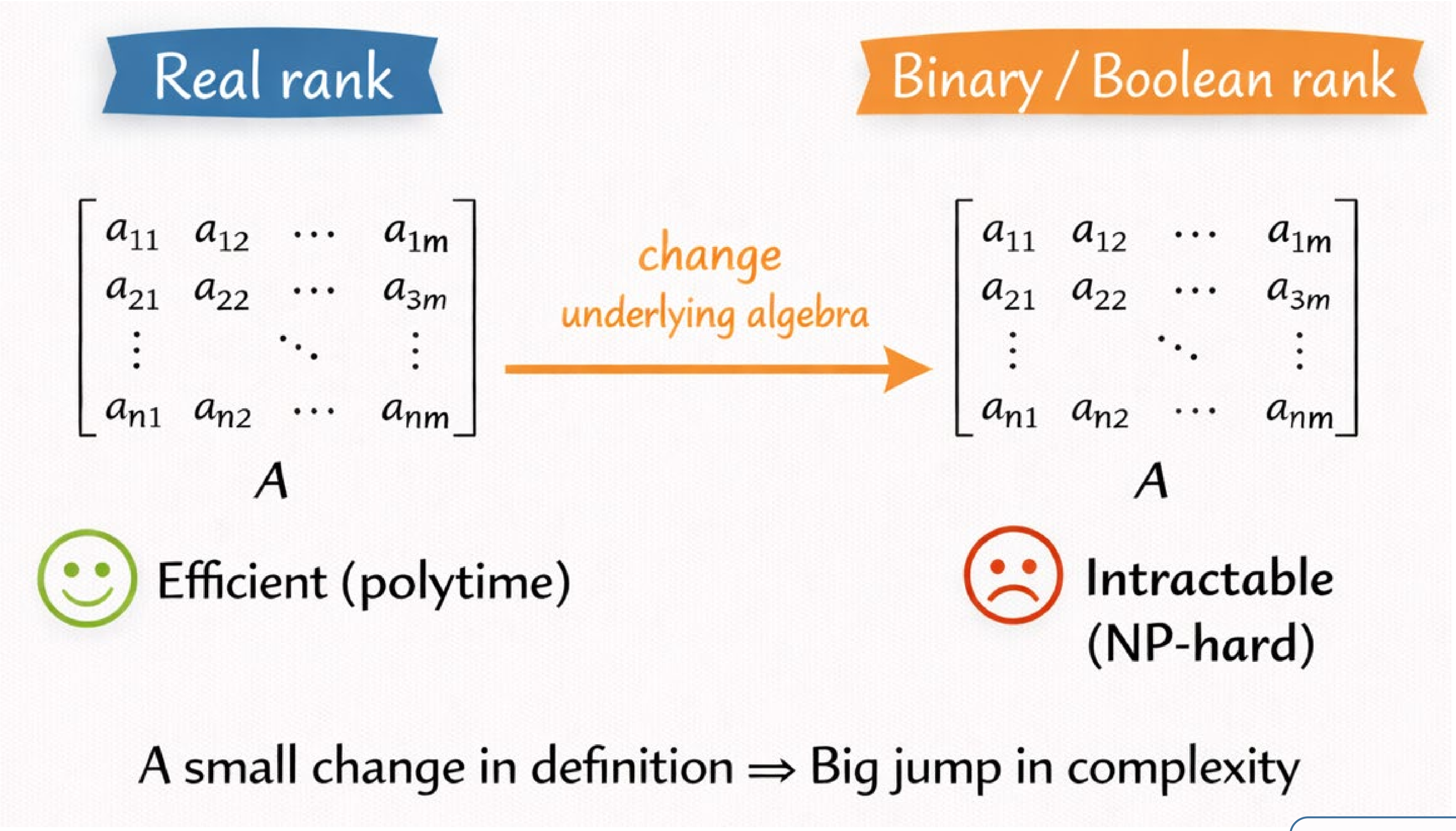
$$D_{n,k} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Real($D_{2k,k}$) = k+1

Boolean($D_{2k,k}$) = 2k

Binary($D_{2k,k}$) = 2k

Complexity of Rank Functions



Real rank

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{3m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Efficient (polytime)

change underlying algebra

Binary / Boolean rank

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{3m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

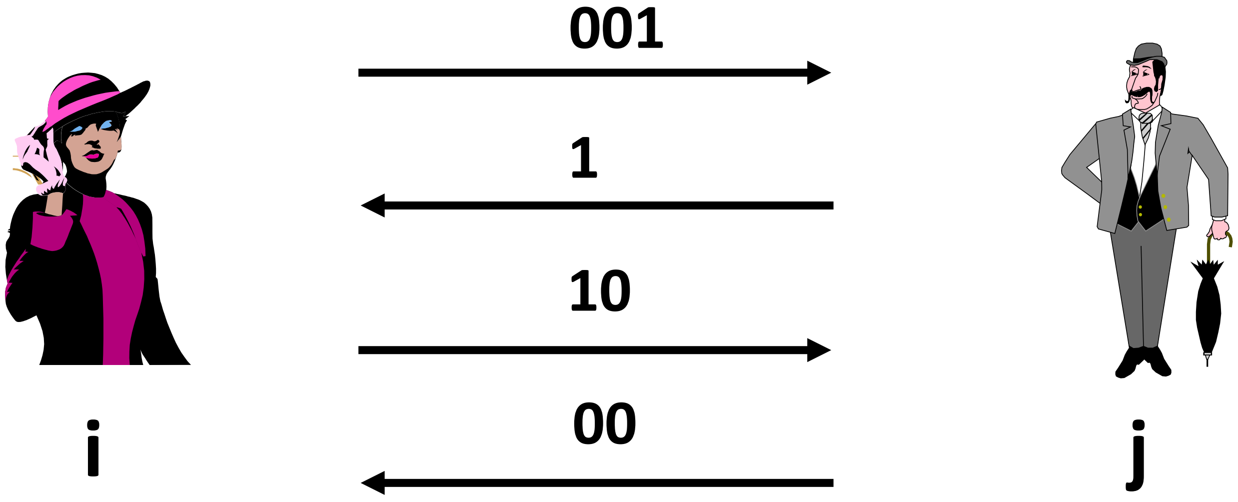
Intractable (NP-hard)

A small change in definition ⇒ Big jump in complexity

- Boolean/binary rank is NP-hard (Stockmeyer 1975, Orlin 1977, Jiang, Ravikumar 1993).
- Boolean rank hard to approximate up to $n^{1-\varepsilon}$ for M_{nxn} (Chalermsook, Heydrich, Holm, Karrenbauer 2014).

Open question: does the same hardness result hold for binary rank?

Communication Complexity



M

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
	0	1	1	1	0	0	0
i	0	0	**0**	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
			j				

Deterministic communication complexity **D(M)**:

minimal #bits communicated to determine $m_{i,j}$

D(M) is NP-hard (Hirahara, Ilango, Loff 2025, Gaspers, He and Mackenzie 2025)

Non-Deterministic Scenario

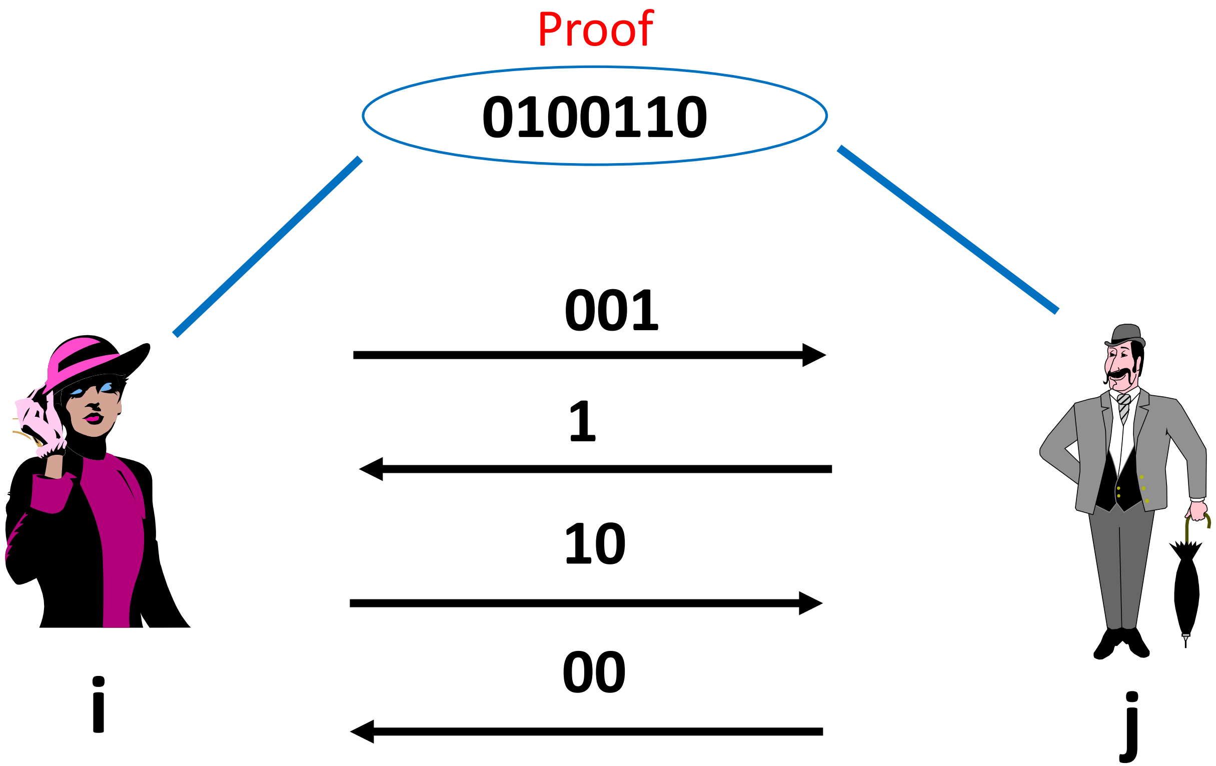
There is a “proof”/certificate such that Alice and Bob accept if and only if $m_{i,j} = 1$.

Non-Deterministic

communication complexity **N(M)**:

minimal #bits communicated

including bits of “proof”.



N(M) is NP-hard (Equivalent to Boolean rank)

Communication Complexity, Binary and Boolean Rank

A protocol induces a partition/cover of the 1's of M.

N(M) = log Boolean(M).

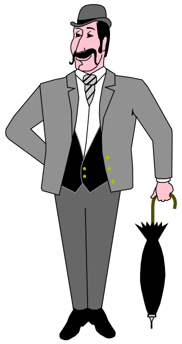
log Binary(M) ≤ D(M) ≤ O($\log^2$ Binary(M))

Yao, 1979

Yannakakis, 1991

M

$$\begin{array}{c|ccccccc}
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\mathbf{i} & 0 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
 & & & \mathbf{j} & & & &
\end{array}$$



Log Rank Conjecture

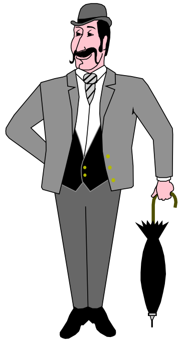
$$\log_2 \mathrm{Real}(M) \le D(M) \le O\left(\sqrt{\mathrm{Real}(M)}\right)$$

Mehlhorn, Schmidt
1982

Sudakov, Tomon
2023

M

$$\begin{array}{c|c|c|c|c|c|c|c|}
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
\mathbf{i} & 0 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 \\ \hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ \hline
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathbf{j}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}
\end{array}$$



Log rank Conjecture (Lovasz, Saks 1988): $\log_2 \mathrm{Real}(M)$ is polynomially related to $D(M)$.

Log Rank Conjecture

M

$$\begin{array}{c|ccccccc}
 & & & & & & & \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
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 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\mathbf{i} & 0 & 0 & \boxed{\mathbf{0}} & 1 & 0 & 1 & 0 \\
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 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
 & & & \mathbf{j} & & & &
\end{array}$$

$$\boxed{\log_2 \mathrm{Real}(M) \le D(M) \le O\left(\sqrt{\mathrm{Real}(M)}\right)}$$

Mehlhorn, Schmidt
1982

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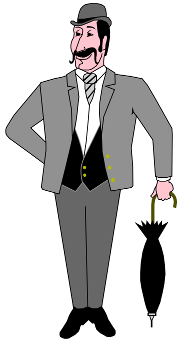
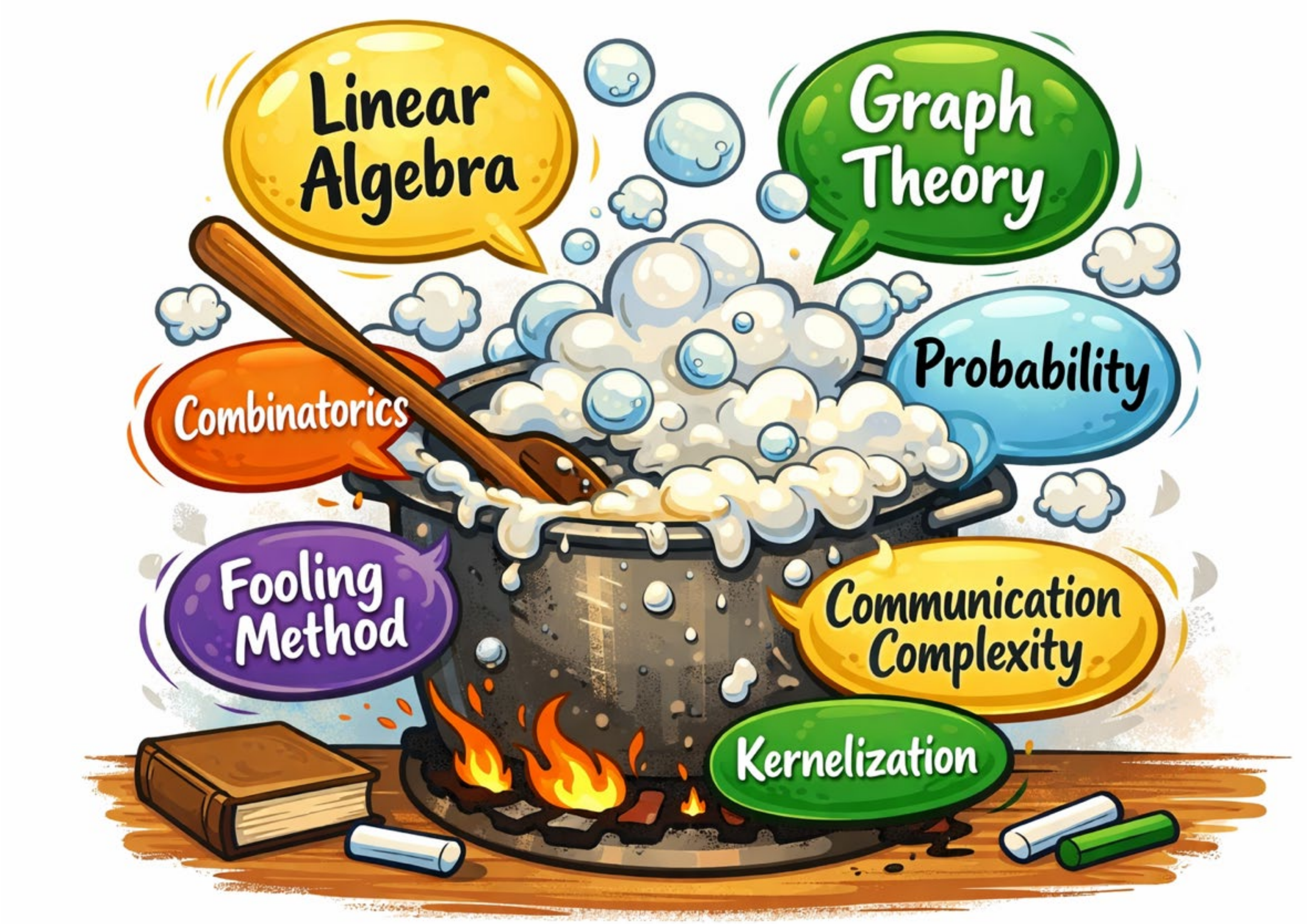


Log rank Conjecture (Lovasz, Saks 1988): $\log_2 \mathrm{Real}(M)$ is polynomially related to $D(M)$.

$$\log \mathrm{Binary}(M) \le D(M) \le O(\log^2 \mathrm{Binary}(M))$$

Equivalent conjecture: log Real(M) is polynmially related to log Binary(M).

Diverse Techniques



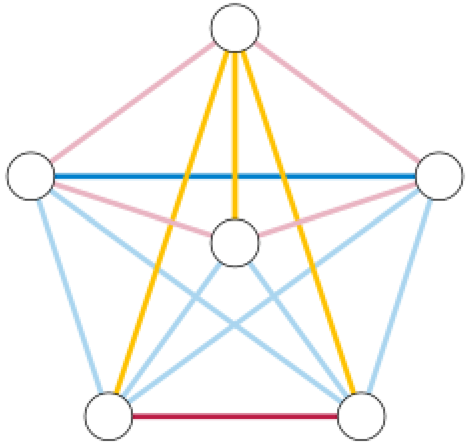
Linear Algebra: Real(M) $\leq$ Binary(M)

The Graham Pollak Theorem, 1971:

Any biclique partition of the edges of K_n

includes at least n-1 bicliques.

All proofs use linear algebra: Graham Pollak, Tverberg 1982, Peck 1984.



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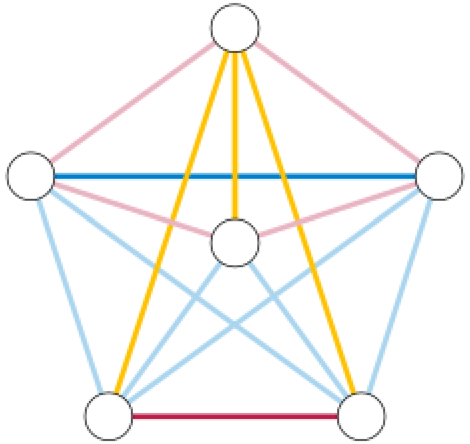

$$D_{6,4} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{Real}(D_{n,k}) = n - \gcd(n,k) + 1$

$\mathrm{Binary}(D_{n,k}) \geq \min\{n, \mathrm{Real}(D_{n,k}) + 1\}$

Gregory 1982 (k = 2),
Haviv, Parnas 2023 (k ≥ 2)

Open question: Better lower bound for Binary($D_{n,k}$)?

Isolation/Fooling sets

A subset of 1 entries of M is an isolation set if no two 1’s are in the same row/column or belong to an all one 2x2 submatrix.

$$\begin{pmatrix} {\color{red}1} & 1 & 1 & 0 & 0 & 0 \\ 0 & {\color{red}1} & 1 & 1 & 0 & 0 \\ 0 & 0 & {\color{red}1} & 1 & 1 & 0 \\ 0 & 0 & 0 & {\color{red}1} & 1 & 1 \\ 1 & 0 & 0 & 0 & {\color{red}1} & 1 \\ 1 & 1 & 0 & 0 & 0 & {\color{red}1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & {\color{red}1} & 1 & 1 & 1 \\ {\color{red}1} & 0 & 1 & 1 & 1 & 1 \\ 1 & {\color{red}1} & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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Claim: No two 1's of an isolation set can be in the same rectangle.

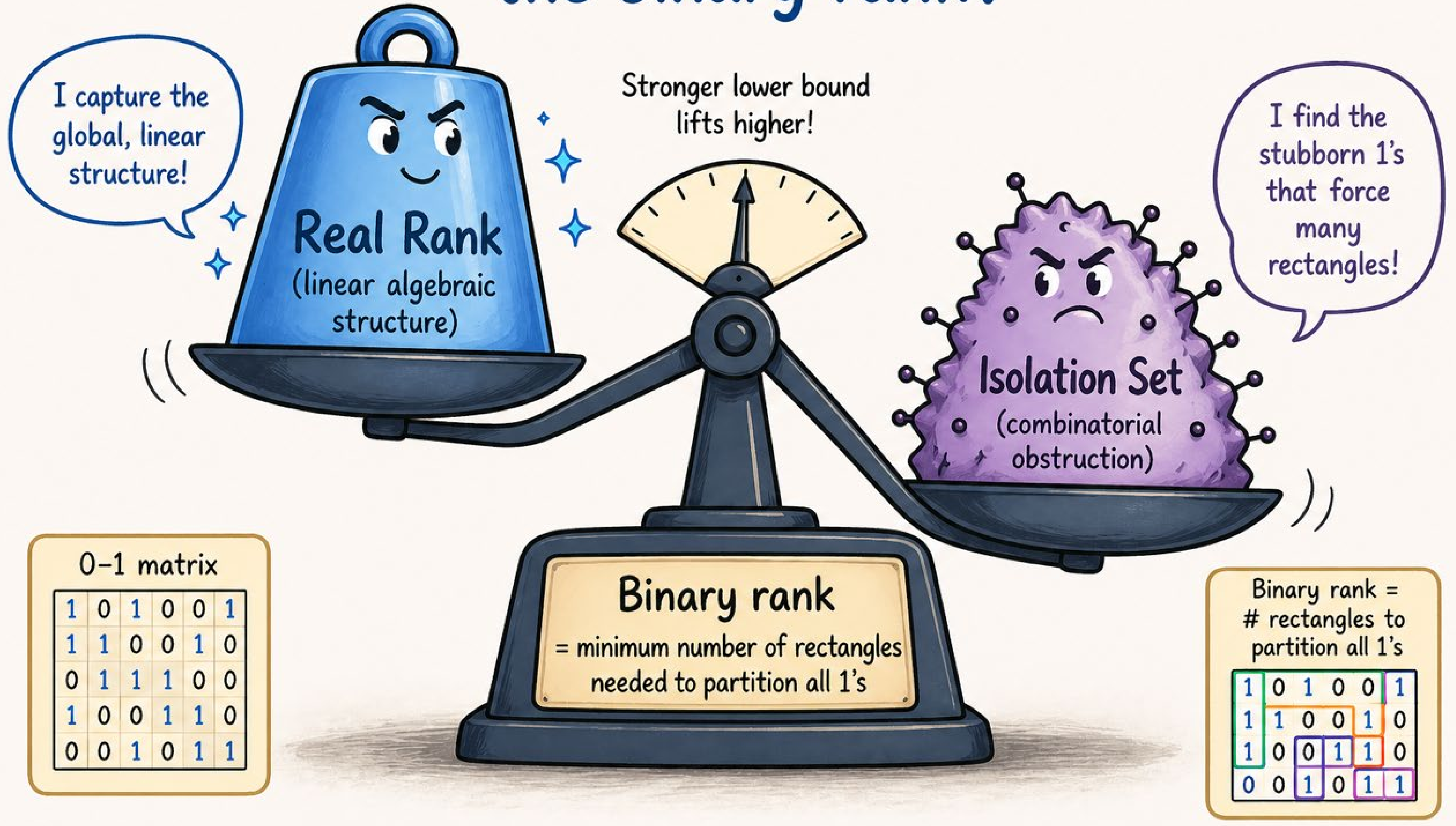
⇓

Maximal Isolation set $\leq$ Minimum rectangle cover/partition.

$$\begin{pmatrix} 0 & 1 & \mathbf{1} & 1 & 1 & 1 \\ \mathbf{1} & 0 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Shitov 2013: Maximal Isolation set is NP-hard.

Which bound is better for the binary rank?



0–1 matrix

1	0	1	0	0	1
1	1	0	0	1	0
0	1	1	1	0	0
1	0	0	1	1	0
0	0	1	0	1	1

Binary rank = # rectangles to partition all 1's

1	0	1	0	0	1
1	1	0	0	1	0
1	0	0	1	1	0
0	0	1	0	1	1

There is no Winner…

$$D_{6,3} = \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 1 \\ 1 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

$\mathrm{Real}(D_{2k,k}) = k + 1$, $\mathrm{Isolation}(D_{2k,k}) = 2k$

$$C_6 = D_{6,1} = \begin{pmatrix} 0 & 1 & \mathbf{1} & 1 & 1 & 1 \\ \mathbf{1} & 0 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\mathrm{Isolation}(D_{n,1}) = 3$, $\mathrm{Real}(D_{n,1}) = n$

There is no Winner…

$$D_{6,3} = \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 1 \\ 1 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

$$C_6 = D_{6,1} = \begin{pmatrix} 0 & 1 & \mathbf{1} & 1 & 1 & 1 \\ \mathbf{1} & 0 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Real($D_{2k,k}$) = k + 1, Isolation($D_{2k,k}$) = 2k

Isolation($D_{n,1}$) = 3, Real($D_{n,1}$) = n

The size of a maximal isolation set ≤ Real(M)^2

Dietzfelbinger, Hromkovic, Schnitger, 1996

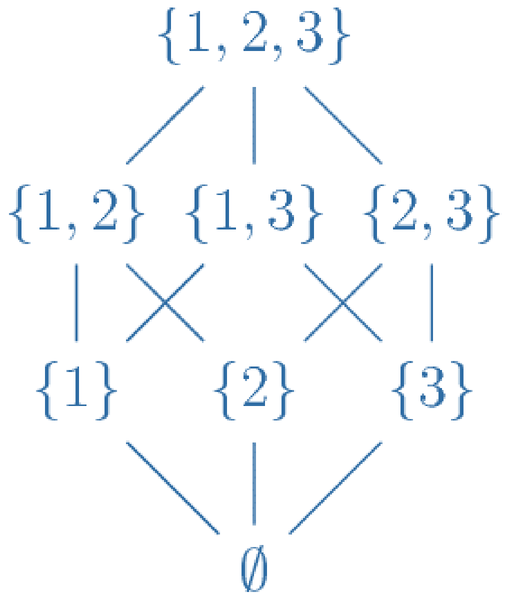
∃ Family of matrices M_{nxn} with Real(M_{nxn}) = $n^{1/2+o(1)}$, Isolation(M_{nxn}) = n

Shigeta, Amano, 2015

Combinatorics: Sperner's Theorem

An anti-chain is a family of sets such that no set is a subset of another.

Sperner, 1928: The size of an anti-chain over {1,2,…,d} is $\leq \binom{d}{\lfloor d/2 \rfloor}$



Combinatorics: Sperner’s Theorem

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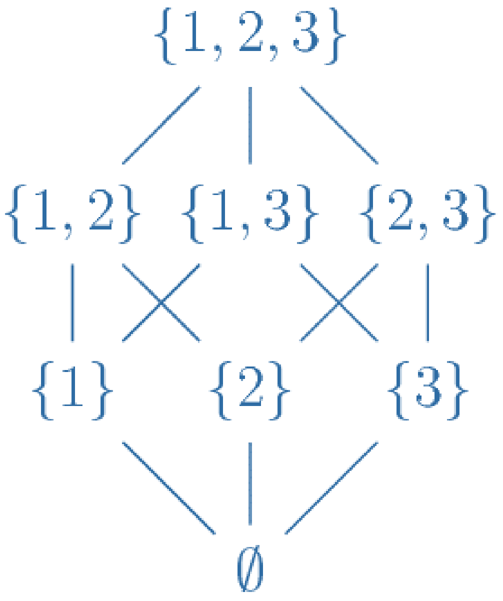


Sperner, 1928: The size of an anti-chain over {1,2,…,d} is $\leq \binom{d}{\lfloor d/2 \rfloor}$

Identify rows of matrix with subsets: (0,1,1,1,1) → {2,3,4,5}

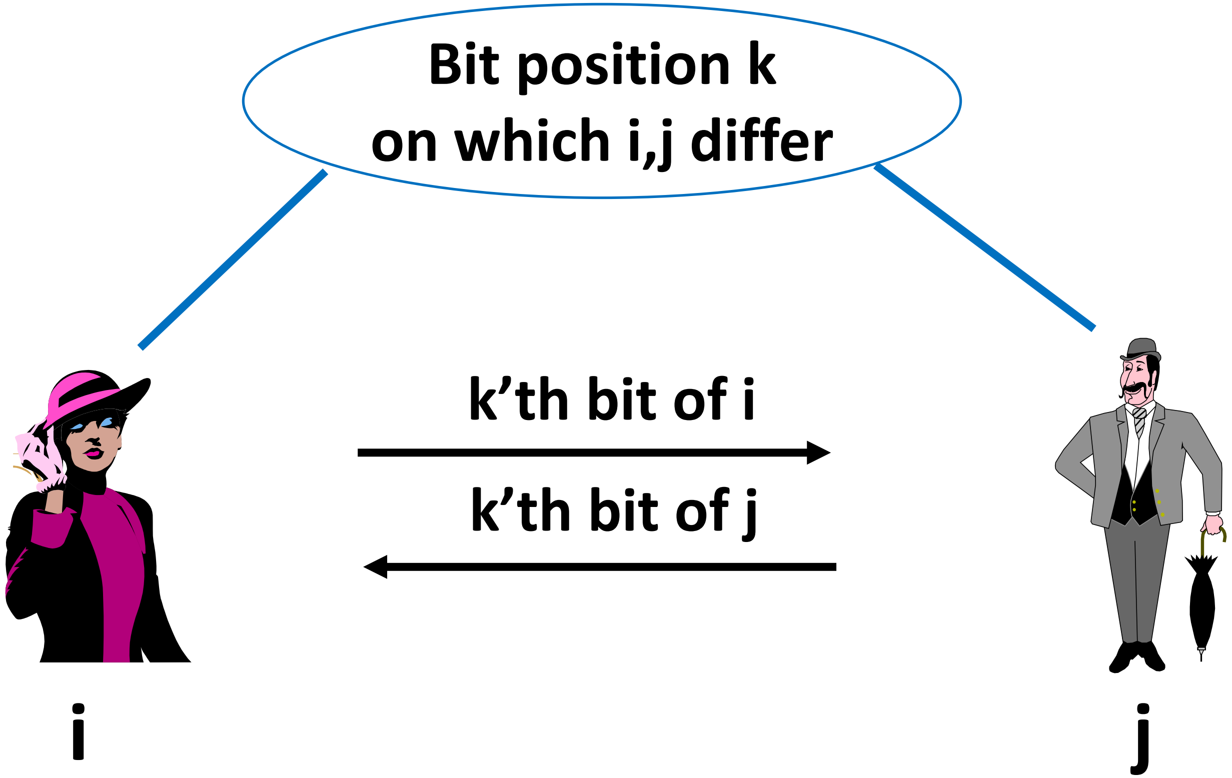
$$C_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\sigma(n) = min \left\{ d \,\middle|\, n \leq \binom{d}{\lfloor d/2 \rfloor} \right\}$$

Boolean(C_n) = $\sigma(n) = \Theta(\log n)$ (de Caen, Gregory and Pullman, 1981)

Communication Complexity Protocols

log Boolean(C_n) = non-deterministic communication complexity.



Bit position k on which i,j differ

k’th bit of i →

k’th bit of j ←

i

j

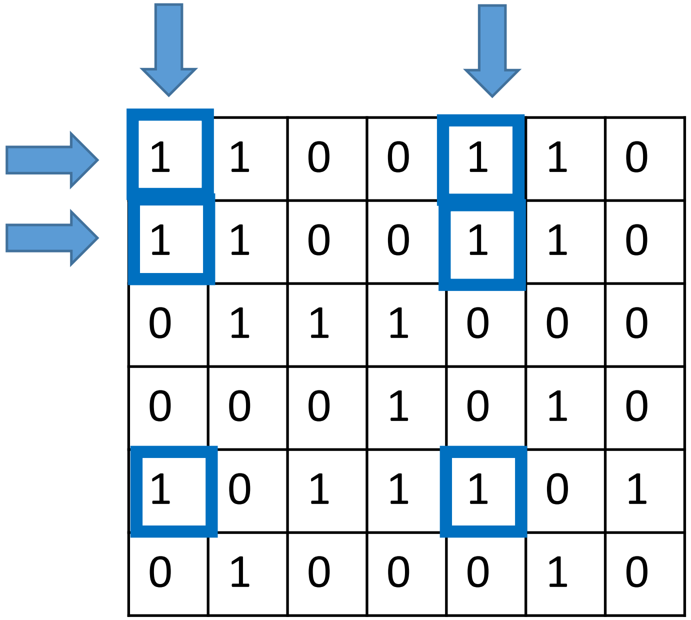
Inequality Matrix: Is $i \neq j$?

$$C_n = \begin{array}{cccccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \\ \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right. & \begin{matrix}0\\1\\1\\1\\1\\1\end{matrix} & \begin{matrix}1\\0\\1\\1\\1\\1\end{matrix} & \begin{matrix}1\\1\\0\\1\\1\\1\end{matrix} & \begin{matrix}1\\1\\\boxed{1}\\0\\1\\1\end{matrix} & \begin{matrix}1\\1\\1\\1\\0\\1\end{matrix} & \begin{matrix}1\\1\\1\\1\\1\\0\end{matrix} & \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right) \begin{matrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\\\mathbf{4}\\\mathbf{5}\\\mathbf{6}\end{matrix} \end{array}$$

No. bits = 2 + loglog n → Boolean(C_n) $\leq 2^{2+loglogn} = O(\log n)$

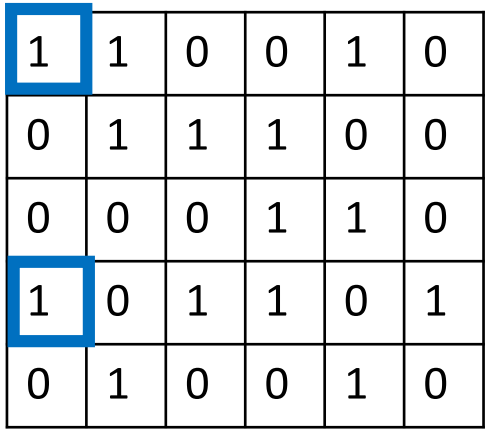
Kernelization

The **kernel** of M is achieved by removing duplicate and all-zero rows/columns.



1	1	0	0	1	1	0
1	1	0	0	1	1	0
0	1	1	1	0	0	0
0	0	0	1	0	1	0
1	0	1	1	1	0	1
0	1	0	0	0	1	0

M



1	1	0	0	1	0
0	1	1	1	0	0
0	0	0	1	1	0
1	0	1	1	0	1
0	1	0	0	1	0

The kernel of M

The kernel of M determines its rank.

An upper bound on Kernel Size

Lemma: #distinct rows/columns of 0,1 matrix M is at most:

$2^{\mathrm{Real(M)}}$ $2^{\mathrm{Binary(M)}}$ $2^{\mathrm{Boolean(M)}}$

An upper bound on Kernel Size

Lemma: #distinct rows/columns of 0,1 matrix M is at most:

$2^{\mathrm{Real(M)}}$ $2^{\mathrm{Binary(M)}}$ $2^{\mathrm{Boolean(M)}}$

$$\mathrm{Real(M)} \leq 2^{\mathrm{Boolean(M)}}$$

Tight

$$C_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Boolean(C_n) = $\Theta(\log_2 n)$
Real(C_n) = n

An upper bound on Kernel Size

Lemma: #distinct rows/columns of 0,1 matrix M is at most:

$2^{\mathrm{Real}(M)}$ $2^{\mathrm{Binary}(M)}$ $2^{\mathrm{Boolean}(M)}$

↓

$\mathrm{Real}(M) \leq 2^{\mathrm{Boolean}(M)}$

Tight

$$C_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

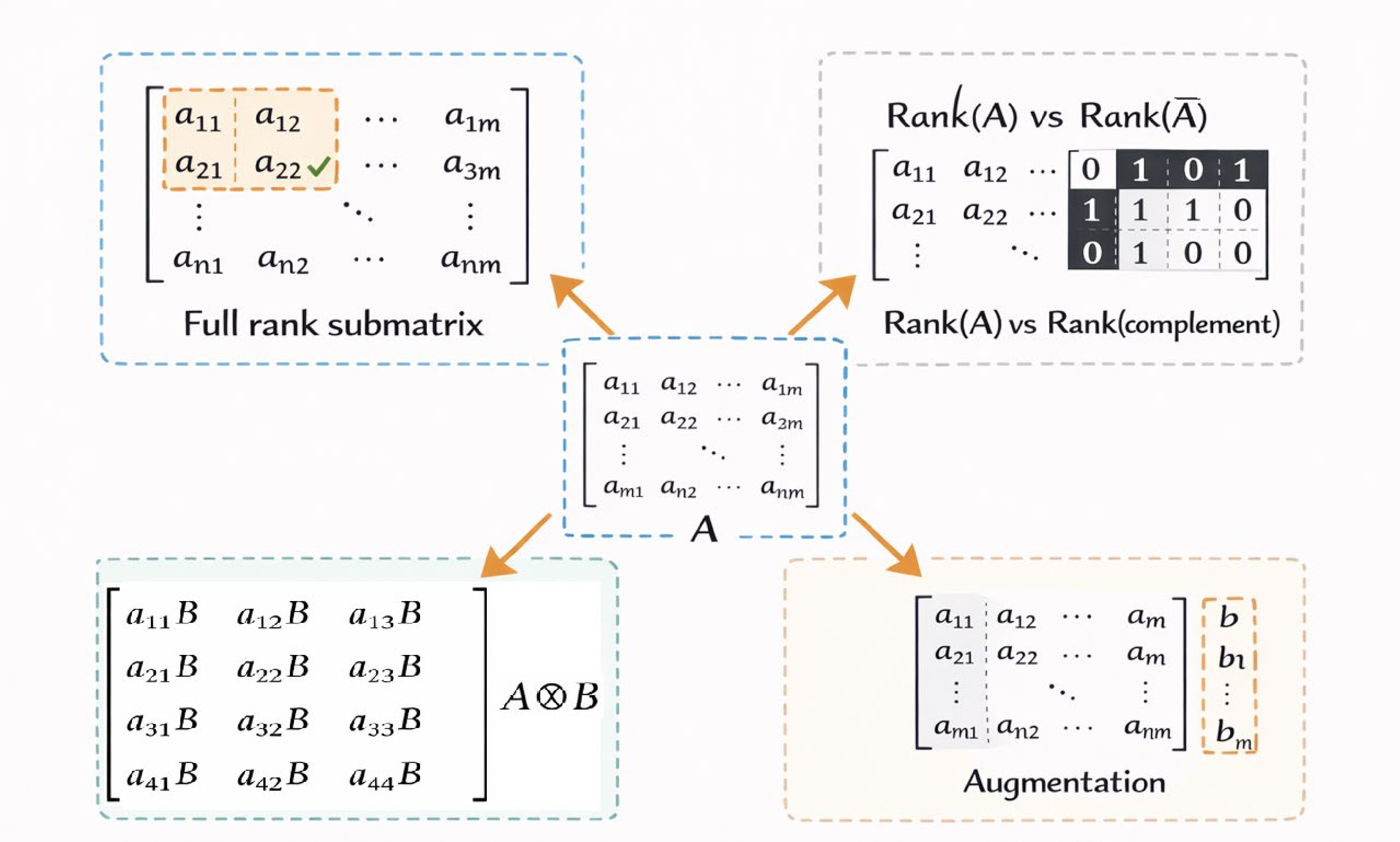
$\mathrm{Boolean}(C_n) = \Theta(\log_2 n)$

$\mathrm{Real}(C_n) = n$

$\mathrm{Binary}(M) \leq 2^{\mathrm{Real}(M)}$

Open question: is this tight?
Log rank conjecture…

Properties of Ranks



Rank(M) vs. Rank($\overline{\mathrm{M}}$)

Definition: $\bar{M} = J - M$, J all 1 matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Claim: For all M, $|\mathrm{Real(M)} - \mathrm{Real}(\overline{\mathrm{M}})| \leq 1$

Proof: $\overline{\mathrm{M}} = \mathrm{J} - \mathrm{M}$, Real(J) = 1.

$$\bar{I}_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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Claim: For all M, $|\mathrm{Real}(M) - \mathrm{Real}(\overline{\mathrm{M}})| \leq 1$

Proof: $\overline{\mathrm{M}} = \mathrm{J} - \mathrm{M}$, Real(J) = 1.

$$\bar{I}_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Definition: M is k-regular if every row and column has k ones.

If M is regular then $\mathrm{Real}(M) = \mathrm{Real}(\overline{\mathrm{M}})$ (Brualdi, Manber, Ross, 1986)

Things change significantly for Boolean and binary rank.

Boolean(M) vs. Boolean($\overline{\mathrm{M}}$)

Claim: Boolean(I_n) = n, Boolean($\bar{I}_n$)= Θ(log n)

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{I}_n = C_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Binary(M) vs. Binary($\overline{\mathrm{M}}$)

Pullman, 1988: Is there a regular M such that Binary(M) $\neq$ Binary($\overline{\mathrm{M}}$)?

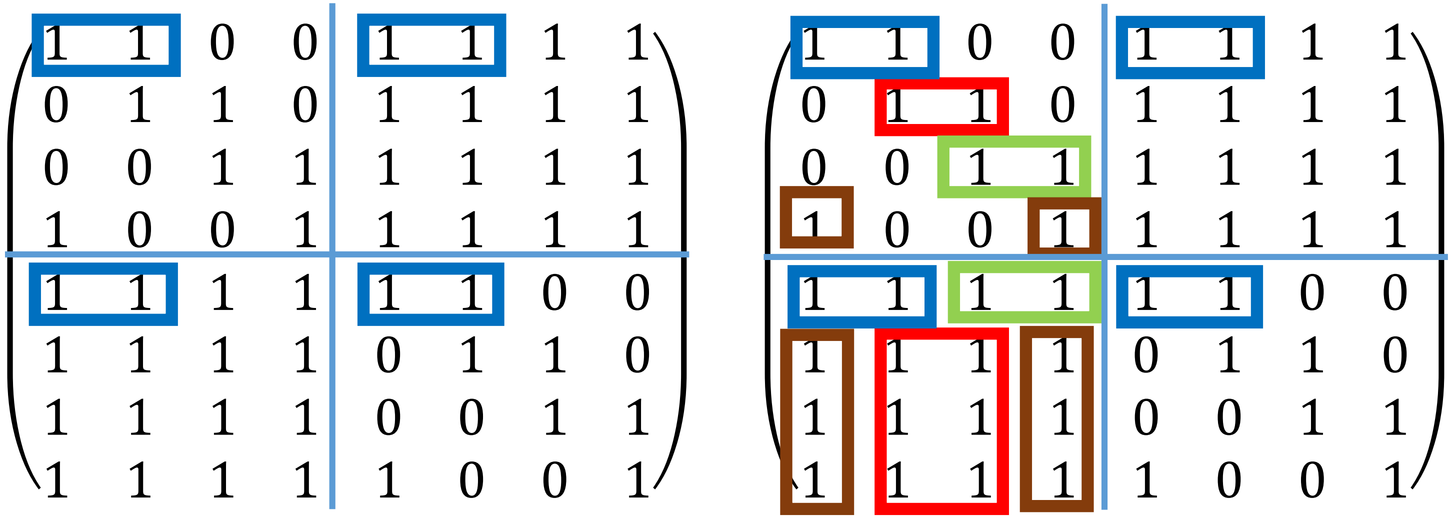
Binary(M) vs. Binary($\overline{\mathrm{M}}$)

Pullman, 1988: Is there a regular M such that Binary(M) ≠ Binary($\overline{\mathrm{M}}$)?

∀k ≥ 2, exists k-regular M such that Binary(M) > Binary($\overline{\mathrm{M}}$). (Haviv, Parnas, 2023)

$$\left(\begin{array}{cccc|cccc}
0 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & 0
\end{array}\right)$$

Isolation set of size 8



$$\left(\begin{array}{cccc|cccc}
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right)
\quad
\left(\begin{array}{cccc|cccc}
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right)$$

Partition of size 7

Largest gap between Binary(M) and Binary($\overline{\mathrm{M}}$)

Recall: $\log \mathrm{Binary}(\mathrm{M}) \leq \mathrm{D}(\mathrm{M}) \leq O(\log^2 \mathrm{Binary}(\mathrm{M}))$

Yannakakis,
1991

Open Problem for many years:
Is upper bound tight?

Largest gap between Binary(M) and Binary($\overline{M}$)

Recall: $\log \text{Binary}(M) \le D(M) \le O(\log^2 \text{Binary}(M))$

Yannakakis, 1991

Open Problem for many years:
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Translates to open problem on Binary(M) vs. Binary($\overline{M}$).

Since: $\log \text{Binary}(\overline{M}) \le D(\overline{M}) = D(M) \le O(\log^2 \text{Binary}(M))$

Largest gap between Binary(M) and Binary($\overline{\mathrm{M}}$)

Recall: log Binary(M) $\leq$ D(M) $\leq$ O($\log^2$ Binary(M))

↑ Yannakakis, 1991

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Translates to open problem on Binary(M) vs. Binary($\overline{\mathrm{M}}$).

Since: log Binary($\overline{\mathrm{M}}$) $\leq$ D($\overline{\mathrm{M}}$) = D(M) $\leq$ O($\log^2$ Binary(M))

Let Binary(M) = d? ➡ ? $\leq$ Binary($\overline{\mathrm{M}}$) $\leq \mathrm{d}^{\mathrm{O}(\log \mathrm{d})}$

Let Binary(M) = d? → ? $\leq$ Boolean($\overline{\mathrm{M}}$) $\leq$ Binary($\overline{\mathrm{M}}$) $\leq \mathrm{d}^{\mathrm{O}(\log \mathrm{d})}$

Boolean($\overline{M}$) =	Authors	Year
$\Omega(d^{6/5})$	Huang and Sudakov	2012
$\Omega(d^{3/2})$	Amano	2014
$\Omega(d^{2})$	Shigeta, Amano	2015
$d^{\Omega(log^{1.28} d)}$	Göös	2015
$d^{\tilde{\Omega}(\log d)}$	Balodis, Ben-David, Göös, Jain, Kothari	2021
$d^{\tilde{\Omega}(\log d)}$	Haviv, Parnas for regular matrices	2023

Combinatorics

Lifting technique

* Results also refute Alon, Saks, Seymour conjecture.

Let Binary(M) = d? $\Rightarrow$? $\leq$ Boolean($\bar{\mathrm{M}}$) $\leq$ Binary($\bar{\mathrm{M}}$) $\leq \mathrm{d}^{\mathrm{O}(\log \mathrm{d})}$

Boolean($\bar{M}$) $=$	Authors	Year
$\Omega\left(d^{6/5}\right)$	Huang and Sudakov	2012
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$d^{\tilde{\Omega}\,(\log d)}$	Haviv, Parnas for regular matrices	2023

Combinatorics

Lifting technique

* Results also refute Alon, Saks, Seymour conjecture.

$\Rightarrow$ $\exists$M, Real(M) $\leq$ d and $\mathrm{Binary}(\mathrm{M}) = d^{\tilde{\Omega}\,(log\ d)}$

Open question: is there a larger gap between real and binary rank?

Full rank sub-matrix

If Real(M) = d then M has sub-matrix of size dxd with real rank d.

$$M = \left(\begin{array}{|cccc|cc} \cline{1-4} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \cline{1-4} \multicolumn{1}{c}{1} & 0 & 0 & \multicolumn{1}{c}{0} & 1 & 1 \\ \multicolumn{1}{c}{1} & 1 & 0 & \multicolumn{1}{c}{0} & 0 & 1 \end{array}\right)$$

Real(M) = 4

Again: Does not hold for binary and Boolean rank!

Boolean(C_7) = 5, every 5x5 sub-matrix has Boolean rank ≤ 4. (de Caen, Gregory and Pullman, 1981)

In fact: Boolean(C_n) = $\Theta(\log_2 n)$ but sub-matrices of size Boolean(C_n) x Boolean(C_n) have an exponentially smaller rank of O(log log n).

$$C_n = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

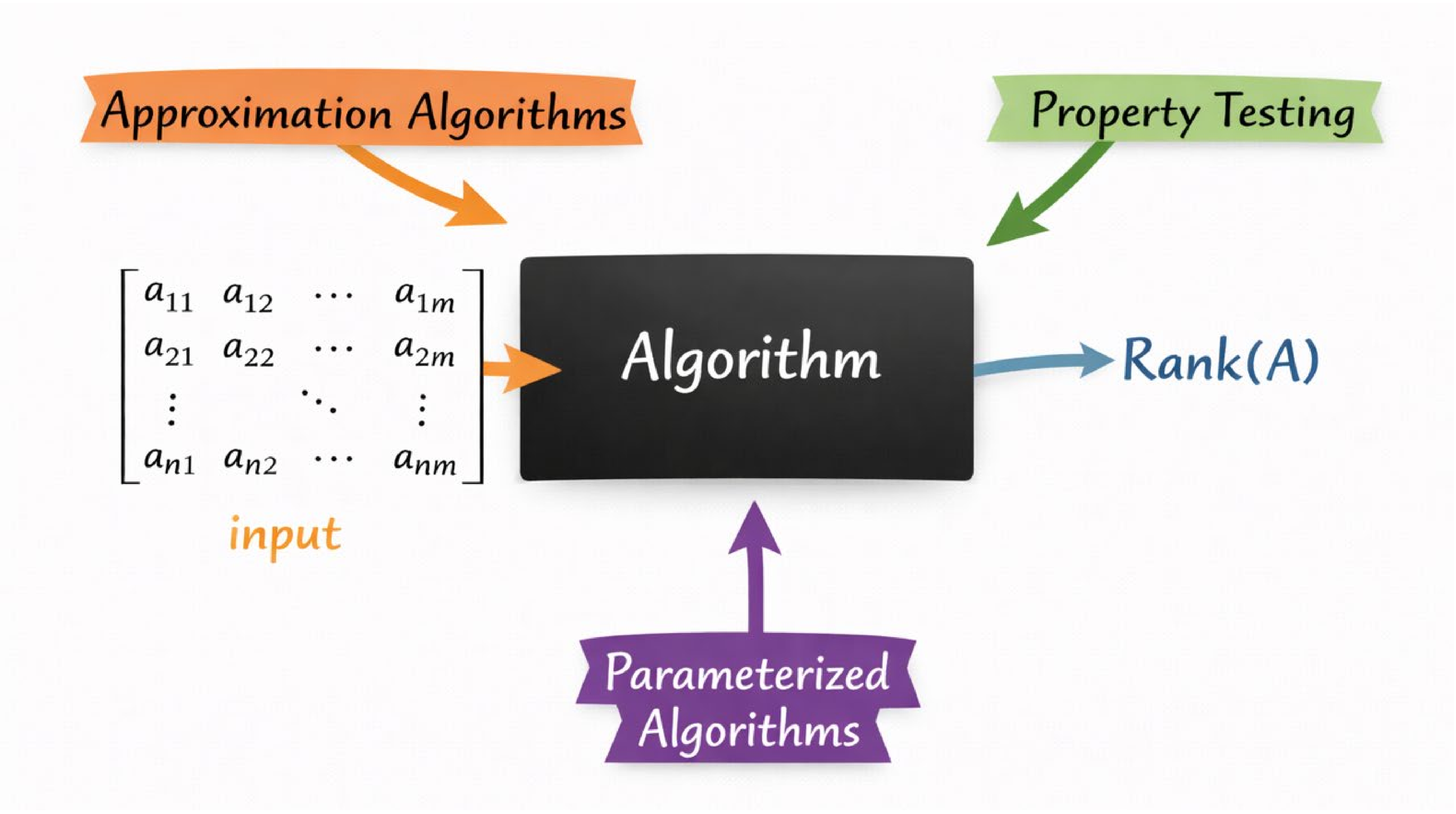
$\exists$ M of size (n+2)x(n+3), n $\geq$ 3, such that:

Binary(M) = n+2 but every (n + 2)x(n + 2) sub-matrix has binary rank $\leq$ n + 1.

Hrubeš, 2025: Binary(M) = 5,
every sub-matrix has rank ≤ 4

Open question: is there an example with a larger gap?

Algorithmic Approaches



Parameterized Algorithms

The Boolean and binary rank are fixed-parameter tractable for the rank d:

∃ algorithms with time Poly(n,m)·f(d), where f is a computable function.

Algorithm: Find small kernel and solve problem on kernel.

Size of kernel
at most 2^dx2^d

Parameterized Algorithms

The Boolean and binary rank are fixed-parameter tractable for the rank d:

∃ algorithms with time Poly(n,m)·f(d), where f is a computable function.

Algorithm: Find small kernel and solve problem on kernel.

Size of kernel
at most 2^dx2^d

Rank	f(d)	Authors
Boolean	$O\left(2^{2^{d2^{d-1}+3d}}\right)$	Nor, Hermelin, Charlat, Engelstadter, Reuter, Duron, Sagot, 2012.
Binary	$O\left(2^{2d^2+d\log d+d}\right)$	Chandran, Issac, Karrenbauer, 2017

Double exponential complexity in d required for Boolean rank unless Exponential Time Hypothesis (ETH) is false.

Approximation Algorithms

Goal: Find Approx such that Rank(A) $\leq$ Approx $\leq$ C·Rank(A), C = Approximation ratio

Boolean rank hard to approximate up to $n^{1-\varepsilon}$ for M_{nxn} (Chalermsook, Heydrich, Holm, Karrenbauer 2014).

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Approximate Boolean/Binary(M_{nxn}**)**

1. Find kernel of M_{nxn}.
2. Cover each row of kernel with a separate rectangle.

Approximation ratio: O(n/log n).

(Chandran, Issac and Karrenbauer, 2017)

Better Approximation Algorithms

Approximate Binary(M_{nxn})

1. Compute Real(M_{nxn}).

2. If Real(M_{nxn}) $\leq \log^2 n/c^4$, output $n^{1/c}$.

3. Else output n.

Approximation ratio: $O(n/\log^2 n)$ (Haviv, 2025)

Proof: $\text{Binary}(M) \leq 2^{D(M)} \leq 2^{c\sqrt{\text{Real}(M)}}$

Better Approximation Algorithms

```
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```

Approximation ratio: $O(n/\log^2 n)$ (Haviv, 2025)

Proof: $\text{Binary(M)} \leq 2^{D(M)} \leq 2^{c\sqrt{\text{Real(M)}}}$

Rank	Approximation ratio	Authors
Boolean	$O\left(\frac{s(\log\log s)^2}{(\log s)^3}\right)$, #1's in matrix = s	Chalermsook, Heydrich, Holm, Karrenbauer, 2014
Boolean	$O(\log m + \log\log n)$, M of size nxm, #1's in column $\leq$ log n.	Miettinen, 2010
Boolean, Binary	At most 2, for Real(M) $\leq$ 4	Parnas, Shraibman 2025

Poly time

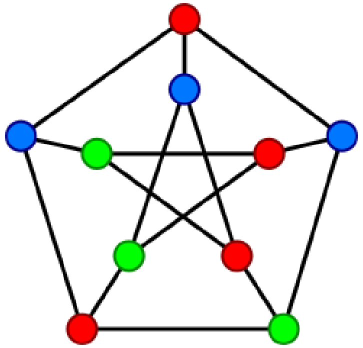
Open question: better approximation algorithms…

Property Testing: Relaxation of Decision Algorithms

Let O be an object (graph, matrix, function)

A Testing Algorithm for property P queries O and:

- Accepts with prob. ≥ 2/3 if O has property P.
- Rejects with prob. ≥ 2/3 if O is "far" from P.



Rubinfeld, Sudan 1996
Goldreich, Goldwasser, Ron, 1998

Goal: Sub-linear query complexity and even independent of object size.

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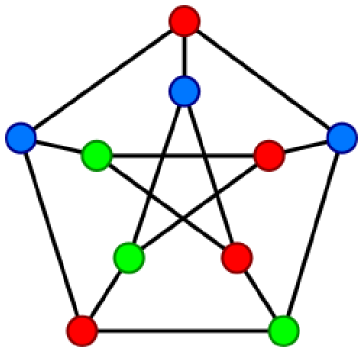
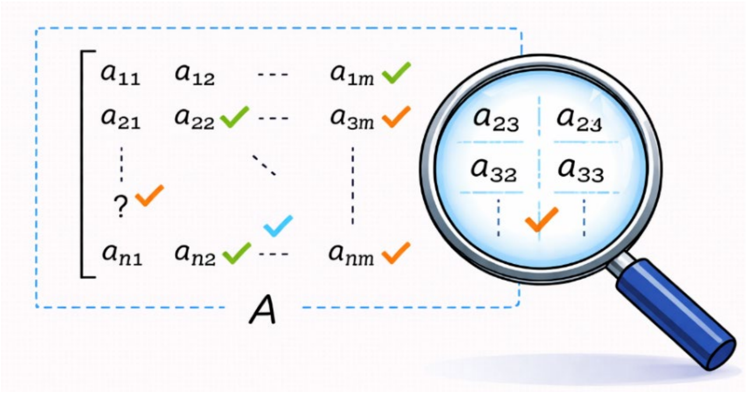


Rubinfeld, Sudan 1996
Goldreich, Goldwasser, Ron, 1998

Goal: Sub-linear query complexity and even independent of object size.

A matrix M is ε-far from rank $\leq$ d if at least an ε-fraction of its entries should be modified so it has rank $\leq$ d



Basic Property Testing Algorithm

Algorithm Test rank(M, d, s**)**

1. Select uniformly, independently, at random s entries $m_{i,j}$

2. If the sxs sub-matrix induced by these entries has rank $\leq$ d then accept. Otherwise, reject.

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} & m_{15} \\ m_{22} & m_{23} & m_{24} & m_{25} & m_{25} \\ m_{32} & m_{33} & m_{34} & m_{35} & m_{35} \\ m_{42} & m_{43} & m_{44} & m_{45} & m_{25} \\ m_{52} & m_{53} & m_{54} & m_{55} & m_{25} \end{pmatrix}$$

Basic Property Testing Algorithm

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Rank	Query Complexity	Authors
Real	$O(d^2/\varepsilon^2)$	Krauthgamer, Sasson, 2003
Boolean	$\tilde{O}(d^4/\varepsilon^6)$	Parnas, Ron, Shraibman, 2021
Binary	$O(2^{2d}/\varepsilon^2)$	Parnas, Ron, Shraibman, 2021
Binary	$\tilde{O}(2^d/\varepsilon^2)$	Bshouty, 2023

Independent of matrix size

Open question: An efficient algorithm for binary rank?

Open Problems

Larger gap between Real and binary rank?

∃M, Real(M) ≤ d and $\text{Binary}(M) = d^{\tilde{\Omega}\,(\log d)}$

Example of a matrix M such that every proper sub-matrix has significantly smaller binary rank.

Better approximation algorithms for Binary/Boolean rank.

Hardness of approximation result for binary rank.

Efficient property testing algorithm for binary rank?

Tight lower bound for Binary($D_{n,k}$).

$$D_{6,4} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

THANK YOU!