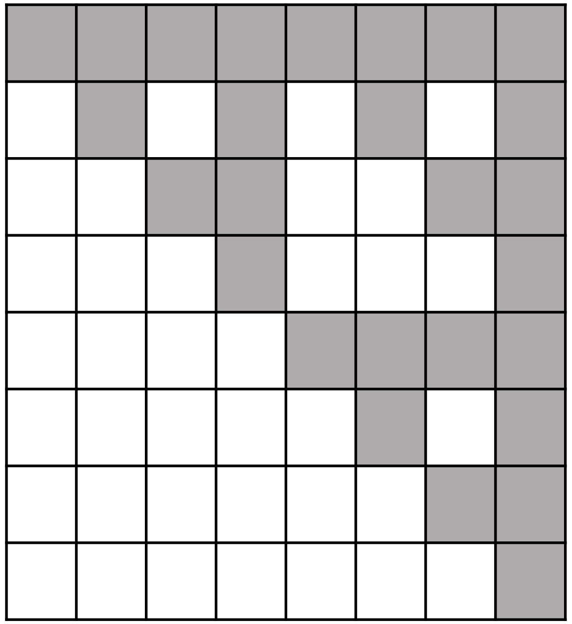
The Boolean and Binary Rank of Kronecker Products: Theory and Applications



Michal Parnas

SAMSA 2026

The Kronecker Product

The Kronecker product $A \otimes B$ of matrices A, B is:

$$A \otimes B = \begin{pmatrix} a_{1,1} \cdot B & \cdots & a_{1,m} \cdot B \\ \vdots & \ddots & \vdots \\ a_{n,1} \cdot B & \cdots & a_{n,m} \cdot B \end{pmatrix}$$

Leopold Kronecker
1823-1891

Georg Zehfuss
1832-1901

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$$A \otimes A = A^{\otimes 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{\text{k times}}$$

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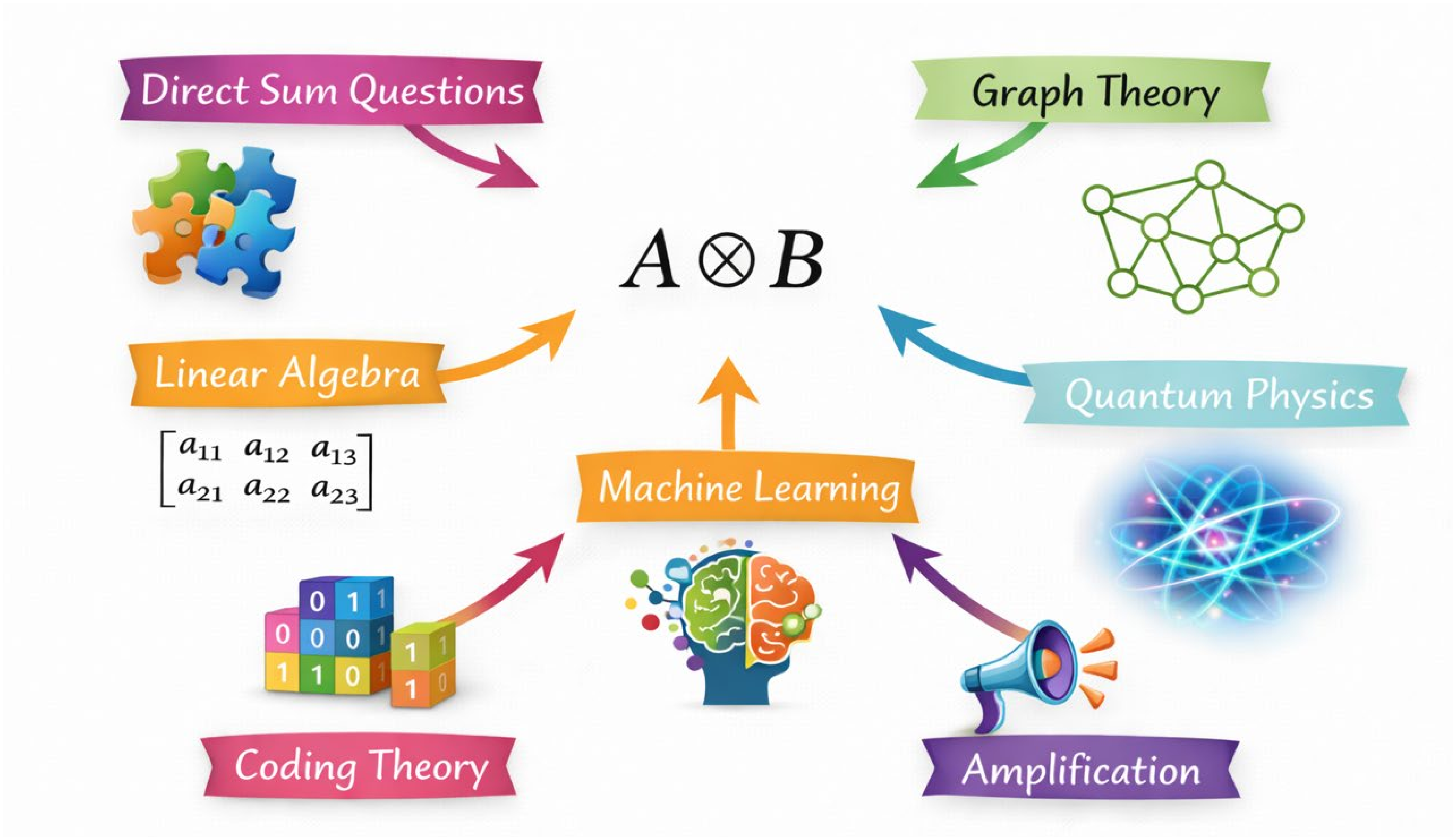
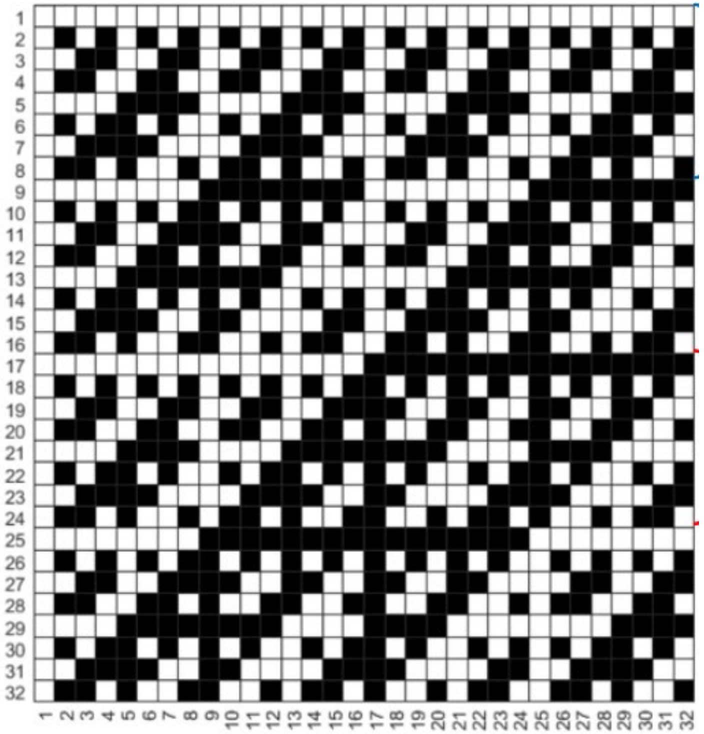
Applications of Kronecker Products



Hadamard Matrices



Square matrix with ± 1 values, where rows are orthogonal.

Used as error correcting codes.

Sylvester's construction, 1867:

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad H_4 = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right) = H_2 \otimes H_2$$

$$H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix} = H_2 \otimes H_{2^{k-1}}$$

The Binary and Boolean Rank

- Binary($M_{n\times m}$) is the minimal d such that: $M_{n\times m} = X_{n\times d} \times Y_{d\times m}$

  $M_{n\times m}$, $X_{n\times d}$, $Y_{d\times m}$ are 0,1 and operations are over the reals.

$$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \cdot \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

- Boolean($M_{n\times m}$): operations are Boolean (1 + 1 = 1).

$$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$$

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  Minimal # monochromatic rectangles to partition all 1’s.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Boolean($M_{n\times m}$): operations are Boolean (1 + 1 = 1).

  Minimal # monochromatic rectangles to cover all 1’s.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The Rank of Kronecker Products

Real rank is multiplicative: $\mathrm{Real}(A \otimes B) = \mathrm{Real}(A) \cdot \mathrm{Real}(B)$

$\text{Binary}(A \otimes B) \leq \text{Binary}(A) \cdot \text{Binary}(B)$

$\text{Boolean}(A \otimes B) \leq \text{Boolean}(A) \cdot \text{Boolean}(B)$

Can there be strict inequality?

$$A \otimes B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ \hline 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ \hline & & & & \end{array}\right)$$

A Lower Bound on the Rank of Kronecker Products

A subset of 1 entries of M is an isolation set if no two 1's are in the same row/column or belong to an all one 2x2 submatrix.

$$\begin{pmatrix} \mathbf{1} & 1 & 1 & 0 \\ 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

Maximal Isolation set $\leq$ Minimum rectangle cover/partition.

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Maximal Isolation set $\leq$ Minimum rectangle cover/partition.

Binary(A $\otimes$ B) $\geq$ max $\{\mathrm{Isolation}(\mathrm{A}) \cdot \mathrm{Binary}(\mathrm{B}), \mathrm{Binary}(\mathrm{A}) \cdot \mathrm{Isolation}(\mathrm{B})\}$

Isolation(A $\otimes$ B) $\geq \mathrm{Isolation}(\mathrm{A}) \cdot \mathrm{Isolation}(\mathrm{B})$

$$A \otimes B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Amplifying Gap using Kronecker products

$$D_{2k,k} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Real($D_{2k,k}$) = k+1,   Isolation($D_{2k,k}$) = Binary($D_{2k,k}$) = 2k

Optimal gap for real rank = 3,4 (Parnas, Shraibman, 2025)

Amplifying Gap using Kronecker products

$$D_{2k,k} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Real($D_{2k,k}$) = k+1, Isolation($D_{2k,k}$) = Binary($D_{2k,k}$) = 2k

Optimal gap for real rank = 3,4 (Parnas, Shraibman, 2025)

Real($D_{4,2}$) = 3, Isolation($D_{4,2}$) = Binary($D_{4,2}$) = 4

Binary($D_{4,2} \otimes D_{4,2}$) $\geq \text{Isolation}(D_{4,2}) \cdot \text{Binary}(D_{4,2}) = 16$

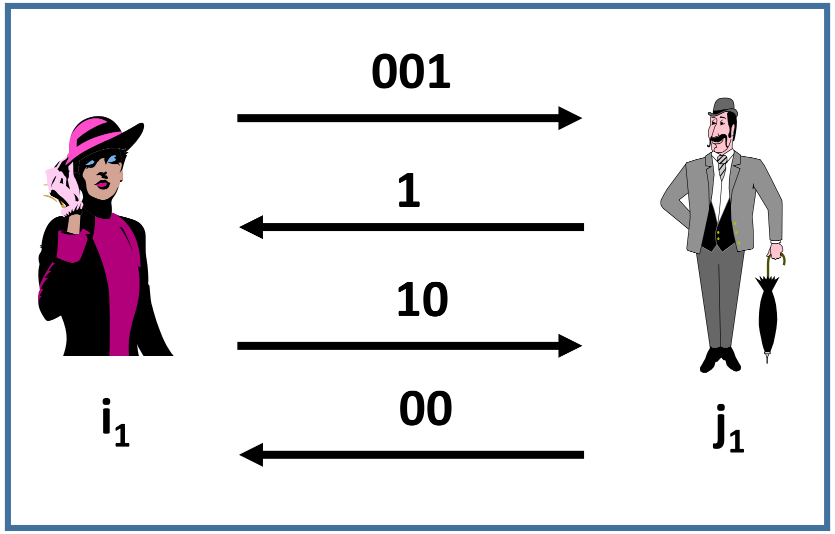
Isolation($D_{4,2} \otimes D_{4,2}$) $\geq \text{Isolation}(D_{4,2}) \cdot \text{Isolation}(D_{4,2}) = 16$

Real$\left(D_{4,2}{}^{\otimes k}\right) = 3^k$, Binary$\left(D_{4,2}{}^{\otimes k}\right)$ = Isolation$\left(D_{4,2}{}^{\otimes k}\right) = 4^k = 2^{2k}$

Direct Sum Problems in Communication Complexity

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
	0	1	1	1	0	1	0
i_1	0	0	**0**	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
			j_1				

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
i_2	0	1	1	1	0	**1**	0
	0	0	0	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
						j_2	



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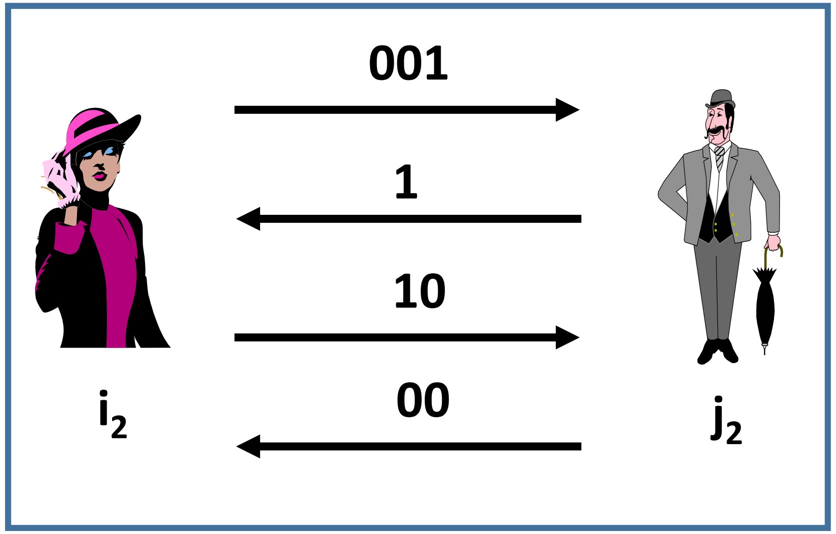
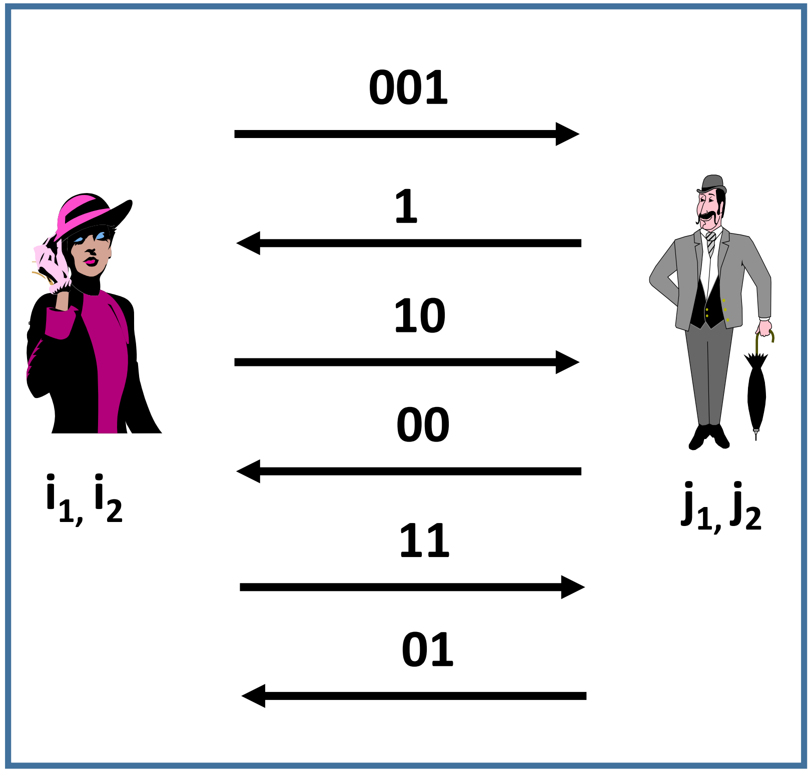


$\geq$



Can there be strict inequality?

Non-Deterministic Scenario

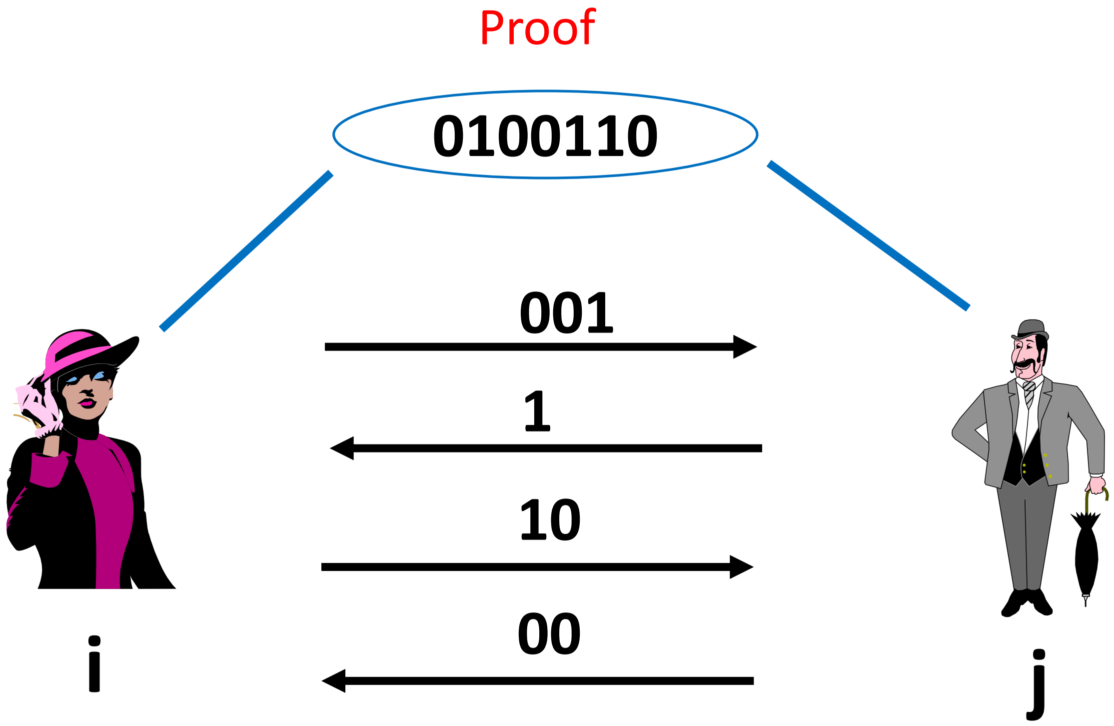
There is a “proof”/certificate such that Alice and Bob accept if and only if $a_{i,j} = 1$.

Non-Deterministic

communication complexity **N(A)**:

minimal #bits communicated

including bits of “proof”.



Direct sum: There is a “proof”/certificate such that Alice and Bob accept if and only if $a_{i_1,j_1} = 1$ and $a_{i_2,j_2} = 1$

Kronecker Products and Direct sum problems

$$A \otimes A = \begin{pmatrix} a_{1,1} \cdot A & \cdots & \cdots & a_{1,m} \cdot A \\ \vdots & \vdots & \vdots & \vdots \\ a_{i_1,1} \cdot A & a_{i_1,j_1} \cdot A & \cdots & a_{i_1,m} \cdot A \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} \cdot A & \cdots & \cdots & a_{n,m} \cdot A \end{pmatrix}$$

$$a_{i_1,j_1} \cdot A = \begin{pmatrix} & & \\ & a_{i_1,j_1} \cdot a_{i_2,j_2} & \\ & & \end{pmatrix}$$

$a_{i_1,j_1} = 1$ and $a_{i_2,j_2} = 1 \Longleftrightarrow a_{i_1,j_1} \cdot a_{i_2,j_2} = 1$

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
	0	1	1	1	0	1	0
i_1	0	0	**0**	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
			j_1				

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
i_2	0	1	1	1	0	**1**	0
	0	0	0	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
						j_2	

Is there Strict Inequality?

$$\boxed{\log \mathrm{Boolean}(A \otimes A) = \mathrm{N}(A \otimes A)}$$

$$\mathrm{N}(A \otimes A) < \mathrm{N}(A) + \mathrm{N}(A)\ ?$$

$$\mathrm{Boolean}(A \otimes A) < \mathrm{Boolean}(A) \cdot \mathrm{Boolean}(A)\ ?$$

$$\begin{array}{c|ccccccc|}
\cline{2-8}
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\mathbf{i_1} & 0 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\cline{2-8}
\multicolumn{1}{c}{} & & & \mathbf{j_1} & & & & \multicolumn{1}{c}{}
\end{array}$$

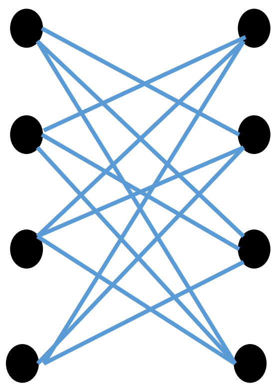
$$\begin{array}{c|ccccccc|}
\cline{2-8}
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\mathbf{i_2} & 0 & 1 & 1 & 1 & 0 & \mathbf{1} & 0 \\
 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\cline{2-8}
\multicolumn{1}{c}{} & & & & & & \mathbf{j_2} & \multicolumn{1}{c}{}
\end{array}$$

A Candidate for Strict inequality

de Caen, Gregory, Pullman, 1988: Conjectured that $\text{Boolean}(C_n \otimes C_n) < \left(\text{Boolean}(C_n)\right)^2$

$$C_4 = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Reduced adjacency matrix of Crown graph.



The inequality function in communication complexity

$$f(i,j) = \begin{cases} 1, & i \neq j \\ 0 & else \end{cases}$$

$$\text{Boolean}(C_n) = min\left\{d \middle| n \leq \binom{d}{\lfloor d/2 \rfloor}\right\} = \Theta(\log n)$$

Lower Bounds

$$\begin{pmatrix} 0 & 1 & \color{red}{1} & 1 \\ \color{red}{1} & 0 & 1 & 1 \\ 1 & \color{red}{1} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\text{Boolean}(C_n \otimes C_n) \geq \text{Isolation}(C_n) \cdot \text{Boolean}(C_n) \geq 3 \cdot \text{Boolean}(C_n)$$

$$\text{Boolean}(A \otimes \mathrm{B}) \geq \frac{(\#1'\text{s in A}) \cdot \text{Boolean}(\mathrm{B})}{\max \#1'\text{s in monochromatic rectangle of A}}$$

(Haviv, Parnas 2022)

$$\Rightarrow \quad \text{Boolean}(C_n \otimes C_n) \geq \frac{n(n-1)}{\left\lceil \frac{n}{2} \right\rceil \cdot \left\lfloor \frac{n}{2} \right\rfloor} \cdot \text{Boolean}(C_n) \geq (4 - o(1)) \cdot \text{Boolean}(C_n)$$

Boolean Rank can have Strict Inequality

Asymptotic results:

- Karchmer, Kushilevitz, Nisan 1995 (Inequality function)
- Error correcting codes (Inequality function)
- Jukna 2022: Probabilistic method (Generalizes Alon 1986)

$\mathrm{Boolean}(C_n \otimes C_n) = \Theta(\log n)$

Combinatorial Results:

Constructive!

- Watts 2001: $\mathrm{Boolean}(C_4 \otimes C_4) < \mathrm{Boolean}(C_4) \cdot \mathrm{Boolean}(C_4)$
- Haviv, Parnas 2022: $\mathrm{Boolean}(C_n \otimes C_m) < \mathrm{Boolean}(C_n) \cdot \mathrm{Boolean}(C_m)$, $n,m \geq 7$.

Resolves question of de-Caen, Gregory, Pullman for all $n \neq 5,6$.

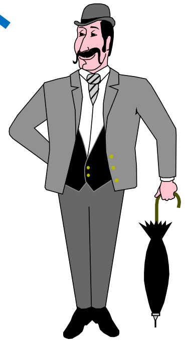
Non Deterministic Protocol for Inequality

Proof

Bit position k
on which i,j differ, bit b

Alice checks if $(i)_k = b$

Bob checks if $(j)_k \neq b$



i

j

Inequality function:

$$f(i,j) = \begin{cases} 1, & i \neq j \\ 0 & else \end{cases}$$

$$C_4 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

bits = 1 + loglog n $\Rightarrow$ Boolean(C_n) $\leq 2^{1+loglogn} = O(\log n)$

Direct Sum of Inequality using Error Correcting Codes

Input: $(i_1, j_1), (i_2, j_2) \in [n] \times [n]$ Output: Accept iff $i_1 \neq j_1$ and $i_2 \neq j_2$

Error correcting code E: Length m = O(log n), alphabet size c = O(1), distance > m/2.

Direct sum of inequality using Error Correcting Codes

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Proof: position k, letters b_1,b_2

Protocol:

Alice checks: $E(i_1)_k = b_1$ and $E(i_2)_k = b_2$

Bob checks: $E(j_1)_k \neq b_1$ and $E(j_2)_k \neq b_2$

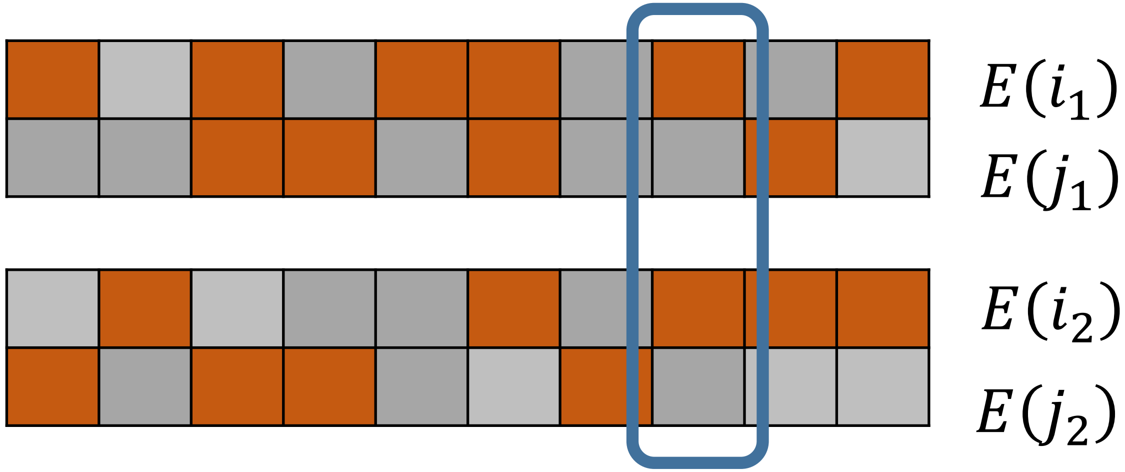
Direct sum of inequality using Error Correcting Codes

Input: $(i_1, j_1), (i_2, j_2) \in [n] \times [n]$ Output: Accept iff $i_1 \neq j_1$ and $i_2 \neq j_2$

Error correcting code E: Length m = O(log n), alphabet size c = O(1), distance > m/2.

Proof: position k, letters b_1,b_2

Distance of code > m/2 ⟹ ∃k on which both pairs differ.



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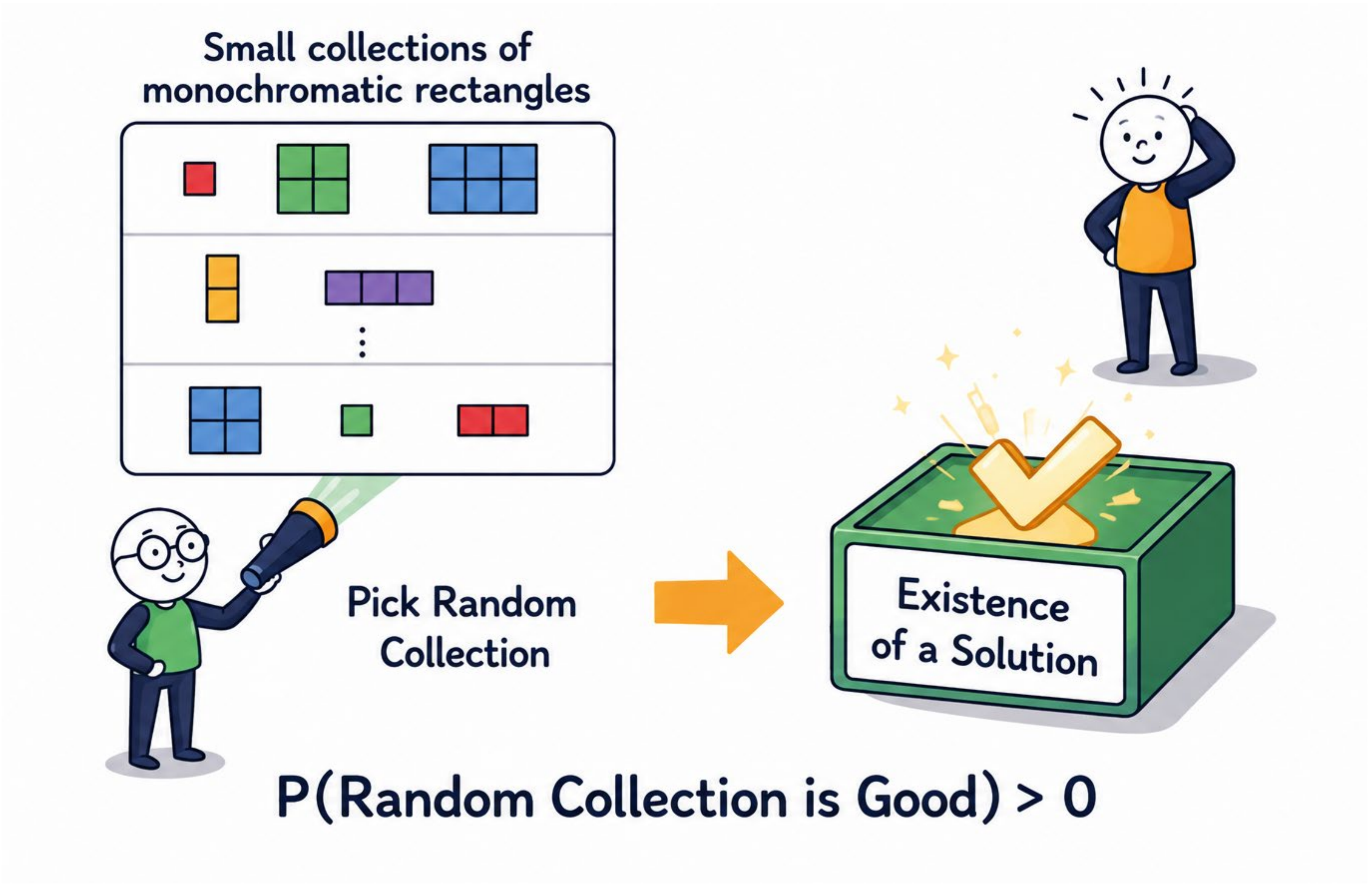
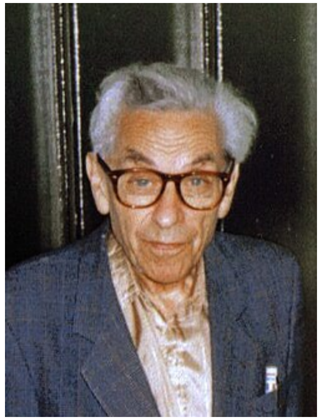
Bob checks: $E(j_1)_k \neq b_1$ and $E(j_2)_k \neq b_2$

bits = O(loglog n), Boolean($C_n \otimes C_n$) = $O(\log n)$

The Probabilistic Method

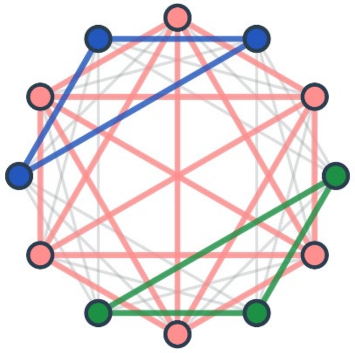




Paul Erdős, 1913-1992

Bounding the Boolean Rank using Probabilistic Method

Alon, 1986 : Upper bound on minimal clique covering of the edges of a graph whose complement has a degree bound.



Theorem: Let $M_{n\times n}$ with $\leq$ d zeros in each column. Then Boolean($M_{n\times n}$) = O(d·log n).

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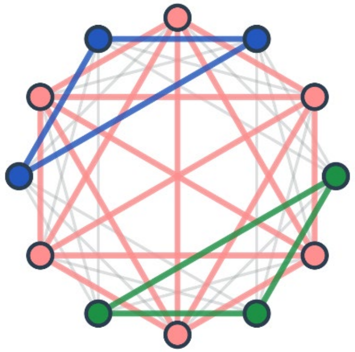

Theorem: Let $M_{n\times n}$ with $\leq$ d zeros in each column. Then Boolean($M_{n\times n}$) = O(d·log n).

Jukna, 2022: Let A, B of size $n \times n$, both with $\leq$ d zeros in each column.

$\Rightarrow$ $\text{Boolean}(A \otimes B) = O(d^2 log n)$.

$\Rightarrow$ $\text{Boolean}(\mathrm{C_n} \otimes \mathrm{C_n}) = \Theta(\log \mathrm{n})$

Proof of Theorem

Show how to select randomly $O(d^2 log n)$ monochromatic rectangles such that with probability > 0 the rectangles cover all 1's of $A \otimes B$.

A:

1	1	0
1	1	1
0	1	1

$\otimes$

B:

1	0	1
0	1	1
0	0	1

=

1	0	1	1	0	1	0	0	0
0	1	1	0	1	1	0	0	0
0	0	1	0	0	1	0	0	0
1	0	1	1	0	1	1	0	1
0	1	1	0	1	1	0	1	1
0	0	1	0	0	1	0	0	1
0	0	0	1	0	1	1	0	1
0	0	0	0	1	1	0	1	1
0	0	0	0	0	1	0	0	1

Random process to choose rows.

Columns added to maximize rectangle with these rows.

Represent row i by a pair (i_1, i_2):

i_1 = index of row of blocks,

i_2 = row index in block.

Row 4 represented by the pair (2,1)

	1			2			3		
1	0	1	1	0	1	0	0	0	
0	1	1	0	1	1	0	0	0	1
0	0	1	0	0	1	0	0	0	
1	0	1	1	0	1	1	0	1	
0	1	1	0	1	1	0	1	1	2
0	0	1	0	0	1	0	0	1	
0	0	0	1	0	1	1	0	1	
0	0	0	0	1	1	0	1	1	3
0	0	0	0	0	1	0	0	1	

Represent row i by a pair (i_1, i_2):

i_1 = index of row of blocks,

i_2 = row index in block.

$\forall\, i = (i_1, i_2)$: choose i_1, i_2 independently and randomly, each with probability $\frac{1}{d+1}$

$\Rightarrow$ $i = (i_1, i_2)$ is chosen with probability $\left(\frac{1}{d+1}\right)^2$

	1			2			3		
1	0	1	1	0	1	0	0	0	
0	1	1	0	1	1	0	0	0	1
0	0	1	0	0	1	0	0	0	
1	0	1	1	0	1	1	0	1	
0	1	1	0	1	1	0	1	1	2
0	0	1	0	0	1	0	0	1	
0	0	0	1	0	1	1	0	1	
0	0	0	0	1	1	0	1	1	3
0	0	0	0	0	1	0	0	1	

Represent row i by a pair (i_1, i_2):

i_1 = index of row of blocks,

i_2 = row index in block.

$$\begin{array}{c|ccc|ccc|ccc|c} & & 1 & & & 2 & & & 3 & & \\ \hline & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \\ & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ \hline & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \\ & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 \\ & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \\ \hline & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \\ & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \end{array}$$

$\forall\, i = (i_1, i_2)$: choose i_1, i_2 independently and randomly, each with probability $\frac{1}{d+1}$

$\Rightarrow$ $i = (i_1, i_2)$ is chosen with probability $\left(\frac{1}{d+1}\right)^2$

R = Row indices i represented by selected pairs (i_1, i_2).

C = Column indices j such that $\forall i \in R$, (i, j) is a 1 entry.

$\mathrm{S} = \mathrm{R} \times \mathrm{C}$ is a monochromatic rectangle.

The probability that a 1 entry (i, j) is covered by rectangle $\mathrm{S} = \mathrm{R} \times \mathrm{C}$ is:

$$\left(\frac{1}{d+1}\right)^2 \cdot \left(1 - \frac{1}{d+1}\right)^d \cdot \left(1 - \frac{1}{d+1}\right)^d \geq \frac{1}{e^2(d+1)^2}$$

$i = (i_1, i_2) \in R$

We didn't choose a pair (i'_1, i'_2)

with a zero in column j.

A, B have $\leq$ d zeros in each column.

$$\begin{array}{c|ccccccccc|}
 & & & j & & & & & & \\
\hline
 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
i & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}$$

Choose randomly $\mathrm{t} = c \cdot e^2(d+1)^2 \log n$ rectangles.

The probability that a 1 entry (i, j) is not covered by t rectangles is at most:

$$\left(1 - \frac{1}{e^2(d+1)^2}\right)^t \le e^{-t \cdot e^{-2} \cdot (d+1)^{-2}} < \frac{1}{n^4}$$

$\mathrm{t} = c \cdot e^2(d+1)^2 \log n$

With probability > 0, at most n^4 ones of $\mathrm{A} \otimes B$ are all covered by these t rectangles.

	1			2			3		
1	0	1	1	0	1	0	0	0	
0	1	1	0	1	1	0	0	0	1
0	0	1	0	0	1	0	0	0	
1	0	1	1	0	1	1	0	1	
0	1	1	0	1	1	0	1	1	2
0	0	1	0	0	1	0	0	1	
0	0	0	1	0	1	1	0	1	
0	0	0	0	1	1	0	1	1	3
0	0	0	0	0	1	0	0	1	

k'th Power of Kronecker Product

Theorem: Let A be of size nxn, with ≤ d zeros in each column.

Then $\mathrm{Boolean}\left(A^{\otimes k}\right) = O\left(k \cdot e^k \cdot (d+1)^k \cdot \log n\right)$

Corollary: $\mathrm{Boolean}\left({C_n}^{\otimes k}\right) = O\left(c^k \log n\right)$, c constant.

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

k'th Power of Kronecker Product

Theorem: Let A be of size nxn, with ≤ d zeros in each column.

Then $\text{Boolean}\left(A^{\otimes k}\right) = O\left(k \cdot e^{k} \cdot (d+1)^{k} \cdot \log n\right)$

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$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Direct sum question:

Non deterministic communication complexity of *k* instances of inequality: $O(k + loglog\ n)$

Amortized non deterministic communication complexity : $\lim_{k\to\infty} \frac{O(k+loglog\ n)}{k} = O(1)$

A Combinatorial Proof Idea

Find low rank matrices $\mathcal{M}$ = $M_1,\ldots,M_s$ and $\mathcal{N}$ = $N_1,\ldots,N_s$ such that $A \otimes B = \sum_{t=1}^{s} M_t \otimes N_t$

$\Rightarrow$ $\mathrm{Boolean}(A \otimes B) = \mathrm{Boolean}(\sum_{t=1}^{s} M_t \otimes N_t) \leq$

$$\sum_{t=1}^{s} \mathrm{Boolean}(M_t \otimes N_t) \leq \sum_{t=1}^{s} \mathrm{Boolean}(M_t) \cdot \mathrm{Boolean}(N_t)$$

How do we find such sequences of matrices?

A Structural Combinatorial Theorem

$\mathcal{M} = M_1,\ldots,M_s$ covers A

if $A = \sum_{t=1}^{s} M_t$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Haviv, Parnas 2022: Let A and B be 0,1-matrices.

Let $\mathcal{M} = M_1,\ldots,M_s$ and $\mathcal{N} = N_1,\ldots,N_s$ be 0,1 matrices such that:

- $\mathcal{M}$ is a cover of A, and
- $\forall\ a_{i,j} = 1$, the matrices N_t for which $(M_t)_{i,j} = 1$ are a cover of B.

Then, $A \otimes B = \sum_{t=1}^{s} M_t \otimes N_t$.

Generalization of result of Watts 2001 for C_4.

Example of $\mathcal{M}, \mathcal{N}$ for $C_4 \otimes C_4$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$M_1 = N_1$

$M_2 = N_2$

$M_3 = N_3$

Each of Boolean rank 2

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

1 is covered by M_1, M_2

Matrices N_1, N_2 cover C_4.

$$\Rightarrow \quad \mathrm{Boolean}(C_4 \otimes C_4) \leq \sum_{t=1}^{3} \mathrm{Boolean}(\mathrm{M_t}) \cdot \mathrm{Boolean}(\mathrm{N_t}) = 3 \cdot 2 \cdot 2 < 16 = (\mathrm{Boolean}(C_4))^2$$

Proof of Theorem: Show that $A \otimes B = \sum_{t=1}^{s} M_t \otimes N_t$

by comparing block by block:

Let $\mathcal{M}$ = $M_1,\ldots,M_s$ and $\mathcal{N}$ = $N_1,\ldots,N_s$:

- $\mathcal{M}$ is a cover of A, and
- $\forall$ $a_{i,j} = 1$, the matrices N_t for which $(M_t)_{i,j} = 1$ are a cover of B.

Proof of Theorem: Show that $A \otimes B = \sum_{t=1}^{s} M_t \otimes N_t$

by comparing block by block:

Let $\mathcal{M} = M_1,\dots,M_s$ and $\mathcal{N} = N_1,\dots,N_s$:

- $\mathcal{M}$ is a cover of A, and
- $\forall\, a_{i,j} = 1$, the matrices N_t for which $(M_t)_{i,j} = 1$ are a cover of B.

$$A \otimes B = \begin{pmatrix} a_{1,1} \cdot B & \cdots & \cdots & a_{1,m} \cdot B \\ \vdots & \vdots & \vdots & \vdots \\ a_{i,1} \cdot B & a_{i,j} \cdot B & \cdots & a_{i,m} \cdot B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} \cdot B & \cdots & \cdots & a_{n,m} \cdot B \end{pmatrix}$$

$a_{i,j} = 0$

$a_{i,j} \cdot B =$ all-0 block.

$\mathcal{M}$ is a cover of A.

$\Rightarrow$ $\forall M_t \in \mathcal{M}$, $(M_t)_{i,j} = 0$

$\Rightarrow$ $\sum_t (M_t)_{i,j} \cdot N_t =$ all-0 block.

$a_{i,j} = 1$

$a_{i,j} \cdot B = B$.

$\exists\, M_t$ such that $(M_t)_{i,j} = 1$.

The matrices N_t for which $(M_t)_{i,j} = 1$ are a cover of B.

Useful Corollary

Find low rank matrices $\mathcal{M}$ = $M_1,\ldots,M_s$ and $\mathcal{N}$ = $N_1,\ldots,N_s$ such that every $[s/2]$ matrices of $\mathcal{M}$ and $\mathcal{N}$ cover A and B, respectively.

$$\Rightarrow \quad A \otimes B = \sum_{t=1}^{s} M_t \otimes N_t$$

$s = 3, [s/2] = 2$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Theorem: $\mathrm{Boolean}(C_n \otimes C_n) < \mathrm{Boolean}(C_n) \cdot \mathrm{Boolean}(C_n)$ for $n \geq 7$.

Proof: Let $d = \mathrm{Boolean}(C_n)$.

Claim: $\exists$ Matrices A_1, A_2, A_3 of rank $1, d-1, d-1$, such that every two of them cover C_n.

$\boxed{\text{Theorem: } \mathrm{Boolean}(C_n \otimes C_n) < \mathrm{Boolean}(C_n) \cdot \mathrm{Boolean}(C_n) \text{ for } n \geq 7.}$

Proof: Let $d = \mathrm{Boolean}(C_n)$.

$\boxed{\text{Claim: } \exists \text{ Matrices } A_1, A_2, A_3 \text{ of rank } 1,\ d-1,\ d-1, \text{ such that every two of them cover } C_n.}$

$\Rightarrow$ C_n is covered by matrices A_2, A_3, A_1 of rank $d - 1$, $d - 1$, 1.

$\Rightarrow$ Let $\mathcal{M} = A_1, A_2, A_3$ and $\mathcal{N} = A_2, A_3, A_1$.

Theorem: $\mathrm{Boolean}(C_n \otimes C_n) < \mathrm{Boolean}(C_n) \cdot \mathrm{Boolean}(C_n)$ for $n \geq 7$.

Proof: Let $d = \mathrm{Boolean}(C_n)$.

Claim: $\exists$ Matrices A_1, A_2, A_3 of rank $1, d-1, d-1$, such that every two of them cover C_n.

⇒ C_n is covered by matrices A_2, A_3, A_1 of rank $d-1, d-1, 1$.

⇒ Let $\mathcal{M} = A_1, A_2, A_3$ and $\mathcal{N} = A_2, A_3, A_1$.

⇒ Using Corollary to Structural Theorem:

$$Boolean(C_n \otimes C_n) \leq 1 \cdot (d-1) + (d-1) \cdot (d-1) + (d-1) \cdot 1$$
$$= d^2 - 1 < d^2 = (Boolean(C_n))^2$$

Claim: d = Boolean(C_n). $\exists$ Matrices A_1, A_2, A_3 of rank $1, d-1, d-1$, where every two cover C_n.

Proof: Assign subsets of $[d]$ to rows/columns, such that subsets intersect $\Longleftrightarrow$ entry is 1.

$$
\begin{array}{c|cccccccccc}
 & \substack{4\\5} & \substack{3\\5} & \substack{3\\4} & \substack{2\\5} & \substack{2\\4} & \substack{2\\3} & \substack{1\\5} & \substack{1\\4} & \substack{1\\3} & \substack{1\\2} \\
\hline
1,2,3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,4 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,4 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,5 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1,4,5 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
2,3,4 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
2,3,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
2,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Rectangle R_t = Entries where subsets intersect on t.

Define: $A_1 = R_1$, $A_2 = R_2 + R_3 + \ldots + R_d$

C_{10} is covered by matrices A_1, A_2 of rank 1, 4.

Shift subsets of A_2 to create A_3 so that $A_1 + A_3$ and $A_2 + A_3$ both cover C_n.

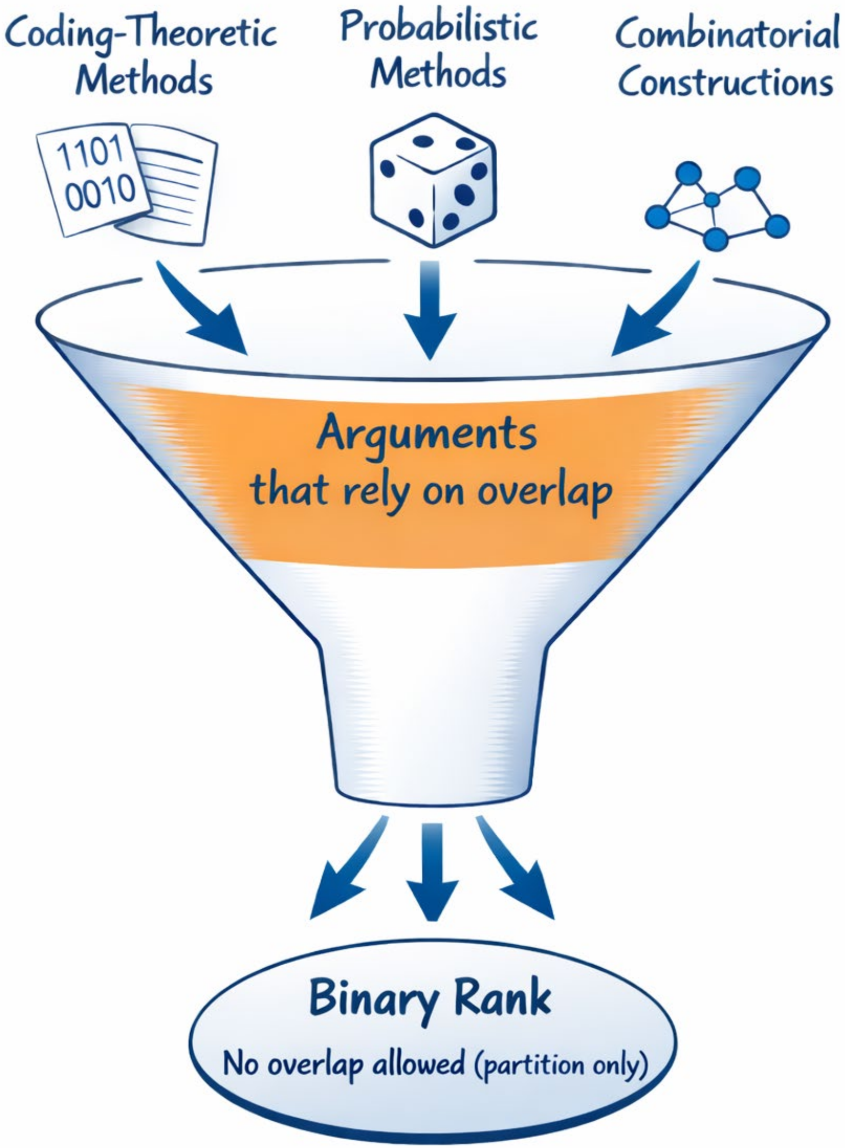
$$
\begin{array}{c|cccccccccc}
 & \substack{4\\5} & \substack{3\\5} & \substack{3\\4} & \substack{2\\5} & \substack{2\\4} & \substack{2\\3} & \substack{1\\5} & \substack{1\\4} & \substack{1\\3} & \substack{1\\2} \\
\hline
1,2,3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,4 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,4 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,5 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1,4,5 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
2,3,4 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
2,3,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
2,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
\quad\Longrightarrow\quad
\begin{array}{c|cccccccccc}
 & \substack{2\\3} & \substack{2\\4} & \substack{2\\5} & \substack{3\\4} & \substack{3\\5} & \substack{4\\5} & \substack{1\\5} & \substack{1\\4} & \substack{1\\3} & \substack{1\\2} \\
\hline
1,4,5 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,5 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,3,4 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,5 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1,2,4 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1,2,3 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
2,3,4 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
2,3,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
2,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3,4,5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Open Question: What about Binary Rank?

$$\text{Binary}(A \otimes B) \leq \text{Binary}(A) \cdot \text{Binary}(B)$$

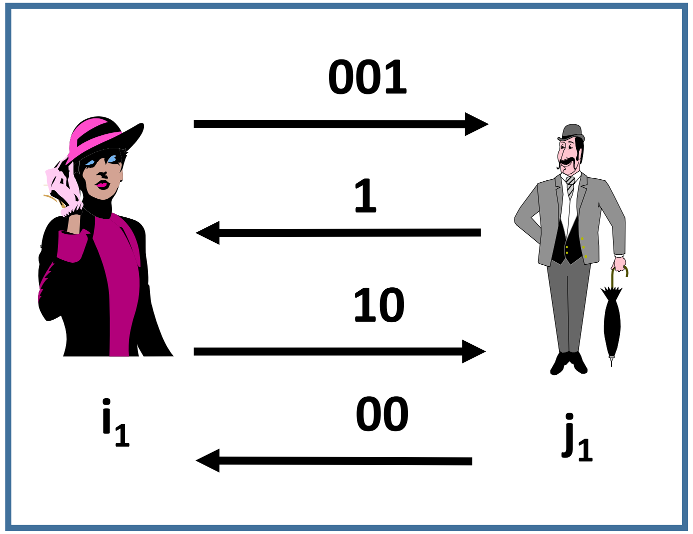
Are there A,B with strict inequality?

All proofs for Boolean rank fail in this case…



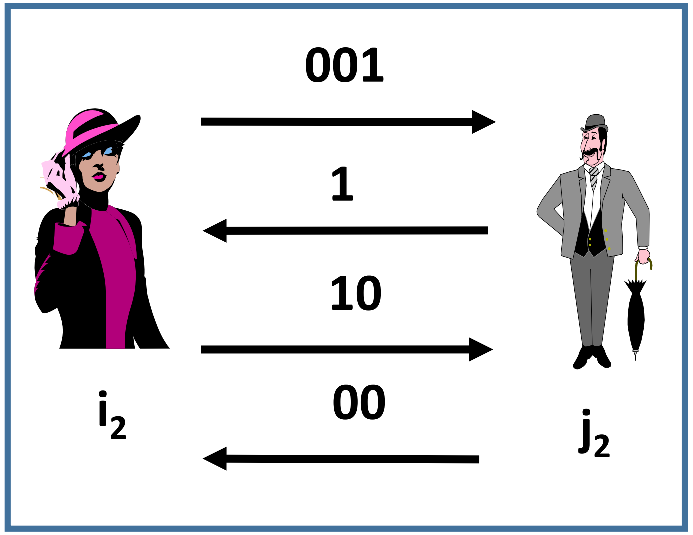
Direct Sum of Deterministic Communication Complexity

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
	0	1	1	1	0	1	0
i_1	0	0	**0**	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
			j_1				



+

	1	1	0	0	1	1	0
	1	1	1	0	0	1	1
i_2	0	1	1	1	0	**1**	0
	0	0	0	1	0	1	0
	1	0	1	1	1	0	1
	0	1	0	0	1	1	0
						j_2	



Mackenzie, Saffidine 2025:

A family of functions for which solving k instances is less than $(1-\varepsilon)k$ times the complexity of solving one instance, for some small $\varepsilon > 0$.

$\boxed{\log \text{Binary}(M) \leq D(M) \leq O(\log^2 \text{Binary}(M))}$

Kronecker Rank of Non-Negative Rank

Non-Negative($M_{n\times m}$) is the minimal d such that $M_{n\times m} = X_{n\times d} \times Y_{d\times m}$

$M_{n\times m}$, $X_{n\times d}$, $Y_{d\times m}$ are non-negative and operations are over the reals.

$$\boxed{\text{Non-Negative}(A \otimes B) \leq \text{Non-Negative}(A) \cdot \text{Non-Negative}(B)}$$

Fawzy 2013: Strict inequality using a computer program.

$$A = \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & 1-a \\ 0 & 0 & 1 & 1-a \\ 1 & 1 & 0 & a \end{pmatrix} \quad a = 3/8$$

$\text{Non-Negative}(A) = 4 \qquad \text{Non-Negative}(A \otimes A) = 15$

Computer Program for Binary Rank?

Parnas, Shraibman, 2025:

Real(A) = 1,2 $\Rightarrow$ Binary(A) = Real(A) $\Rightarrow$ Binary(A$\otimes$A) = Binary(A)$\cdot$ Binary(A).

Real(A) = 3 $\Rightarrow$ Binary(A) $\leq$ 4,

Binary(A) = Real(A) unless A contains *C*, Isolation(*C*) = 4.

$\Rightarrow$ Binary(A$\otimes$A) = Binary(A)$\cdot$ Binary(A).

$$C = \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 1 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

Real(A) = 4 $\Rightarrow$ Binary(A) $\leq$ 6.

At most 7 distinct rows or columns.

Need: Binary(A) $\neq$ Isolation(A), Binary(A) $\neq$ Real(A).

$$M = \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 1 \\ 1 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Real(*M*) = 4
Isolation(*M*) = 5
Binary(*M*) = 6

Non-Negative vs. Binary Rank

By Definition: Real(A) $\leq$ Non-Negative(A) $\leq$ Binary(A)

Watson, 2016: $\exists$ A, Non-Negative(A) $<$ Binary(A)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Amplify gap using Kronecker product?

Non-Negative(A) $=$ 4

Binary(A) $=$ 5

Conjecture: Binary$(A^{\otimes k}) =$ Binary(A)^k

Open Problems

Is binary rank multiplicative under Kronecker product?

Larger gap between Non-negative and binary rank?

Other families of matrices for which Boolean rank is not multiplicative under Kronecker Product?

Use Structural theorem on other families of matrices.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

THANK YOU!