The Sum of Square Roots Problem

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The Sum of Square Roots Problem

SSR

Input: Positive integers $1 \leq a_1, \ldots, a_n \leq 2^n$ and $\delta_1, \ldots, \delta_n \in \{\pm 1\}$

Output: $\operatorname{sgn}\left(\sum_{i=1}^{n} \delta_i \sqrt{a_i}\right)$

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- Fundamental primitive in Computational Geometry.
- Approximate the square roots - what is the precision required?
- Example (due to Ron Graham):
  $\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087} +$
  $\sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198} - \sqrt{1000002} -$
  $\sqrt{1000018} - \sqrt{1000042} - \sqrt{1000066} - \sqrt{1000113} - \sqrt{1000116} -$
  $\sqrt{1000169} - \sqrt{1000175} - \sqrt{1000199} < 10^{-34}$

Separation Bounds

- The minimal polynomial of $\alpha = \sum_{i=1}^{n} \delta_i \sqrt{a_i}$ has degree $d \leq 2^n$ (since $[\mathbb{Q}[\sqrt{a_1}, \ldots, \sqrt{a_i}] : \mathbb{Q}[\sqrt{a_1}, \ldots, \sqrt{a_{i-1}}] \leq 2$).

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- If $\alpha \neq 0$, then

$$\begin{aligned} N(\alpha) &= \prod_{\sigma \in Gal(\mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})/\mathbb{Q})} \sigma(\alpha) \\ &\leq |\alpha| \cdot |\sigma(\alpha)|^{2^n - 1} \\ &\leq |\alpha| \cdot \left(n \cdot \max_{i \in [n]} |\sqrt{a_i}|\right)^{2^n - 1} \\ &\leq |\alpha| \cdot \left(n \cdot 2^{n/2}\right)^{2^n - 1} \end{aligned}$$

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- *Root separation bound (Mignotte'82).* If $p(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x]$, then we have

$$|r_1 - r_2| \geq \frac{\sqrt{3}}{d^{d/2+1}(dH)^{d-1}}$$

where $H = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Prior Work

- **Tiwari (1992)** SSR has an efficient algorithm in the real RAM model of computation (supports unit cost arithmetic): approximate each $\sqrt{a_i}$ up to $2^{O(n)}$ bits via n steps of Newton's method:

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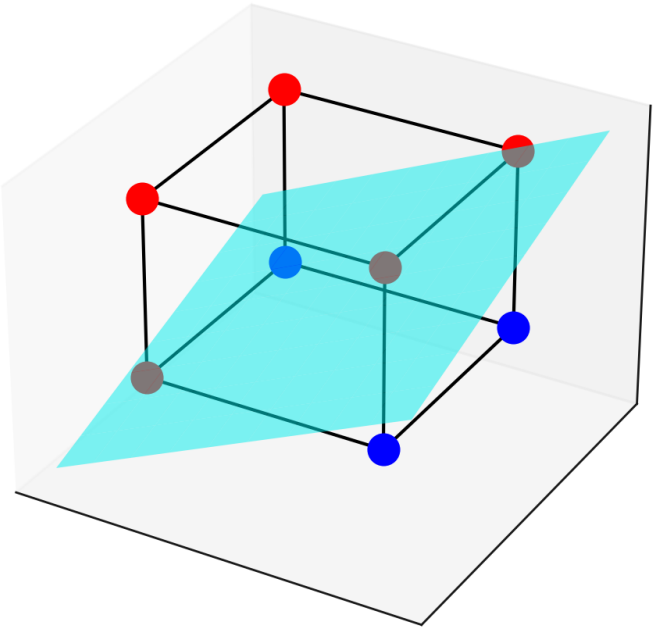
- **SSR-hardness:** Feasibility of Semidefinite Programs (Goemans'98), Approximating 3-player Nash Equilibria (Etessami-Yannakakis'07), Solving systems of monotone polynomial systems (Esparaza-Kiefer-Luttenberger'08).

A different point of view



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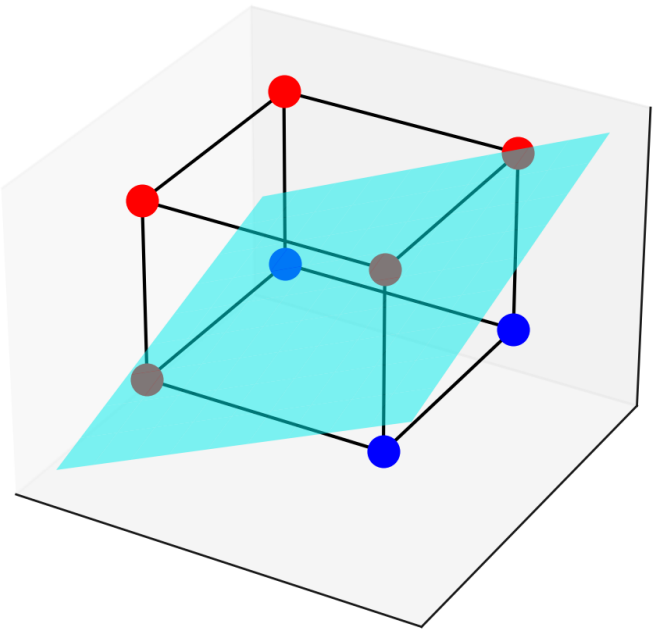

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LTF (Perceptrons)

A Boolean function $f : \{\pm 1\}^n \to \{\pm 1\}$ is a Linear Threshold Function (LTF) if $\exists w_1, \ldots, w_n, \theta \in \mathbb{R}$ such that $\forall x \in \{\pm 1\}^n$

$$f(x) = 1 \iff \sum_{i=1}^{n} w_i x_i - \theta > 0$$

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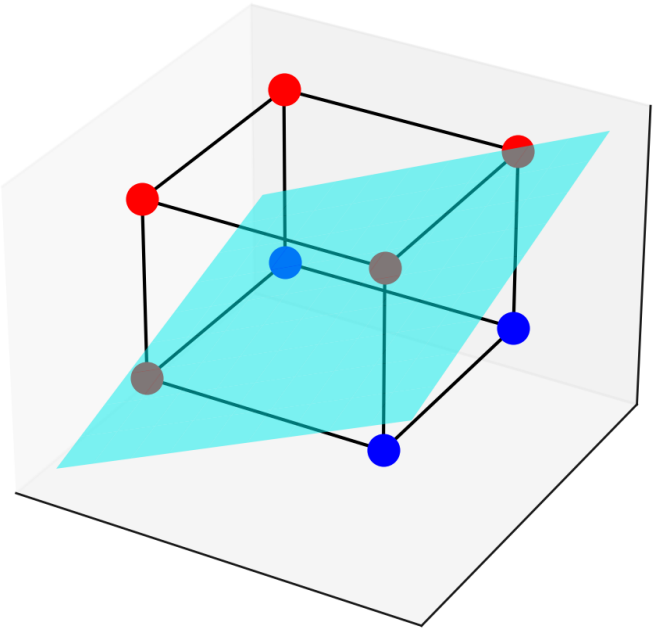

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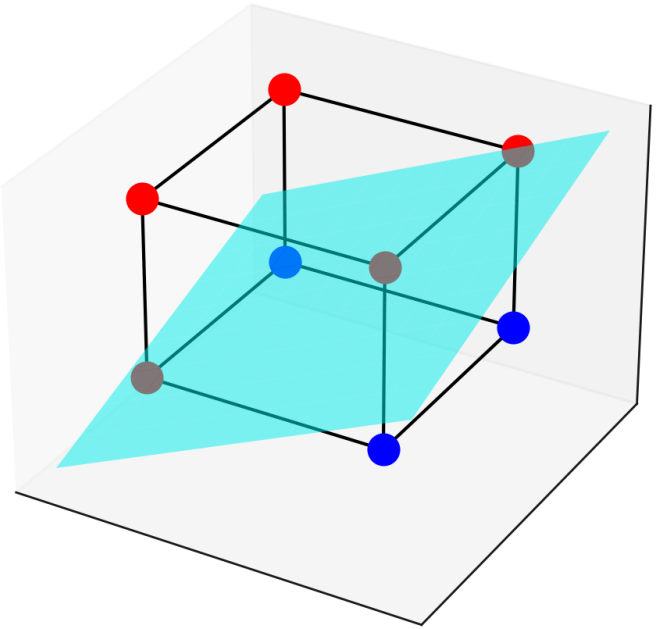

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Do you need real numbers to define an LTF?

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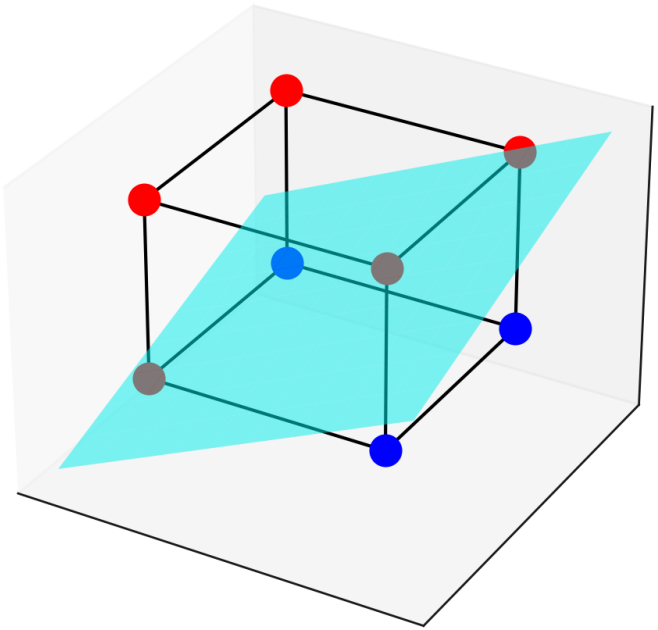

- Let $\varepsilon = min_{x \in \{\pm 1\}^n}(\langle w, x \rangle - \theta) \neq 0$
- Consider for all i, $(\frac{w_i - \varepsilon}{10n}, \frac{w_i + \varepsilon}{10n})$: if you pick w_i' in this interval, then $\forall x \in \{\pm 1\}^n$, $sgn\left(\sum_{i=1}^n w_i x_i - \theta\right) = sgn\left(\sum_{i=1}^n w_i x_i - \theta\right)$ (why?)
- But how small can ε be? If w_i are algebraic, again depends on root separation bounds!
- Can we have w_i' with the above property but with small bit-length (e.g., $poly(n)$ bits?)

Theorem (Muroga'71)

There exist $w_i' \in \mathbb{Z}$ of magnitude at most $2^{O(n \log n)}$ (i.e., representable in $O(n \log n)$ bits)

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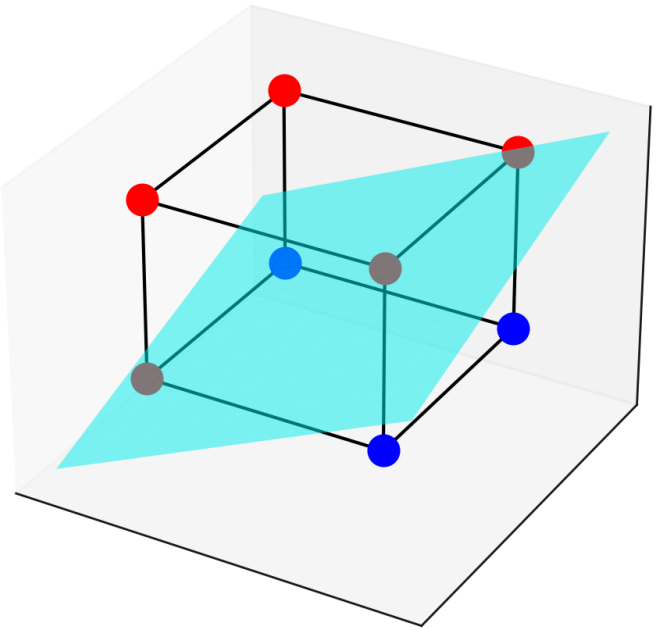

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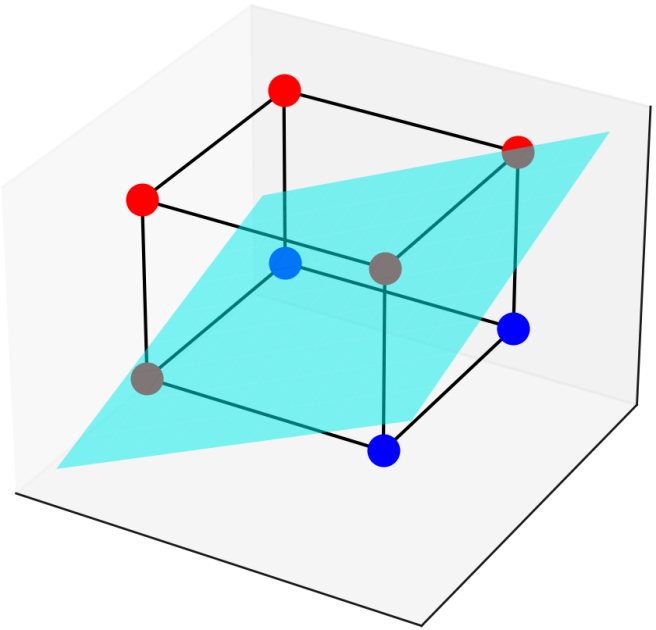

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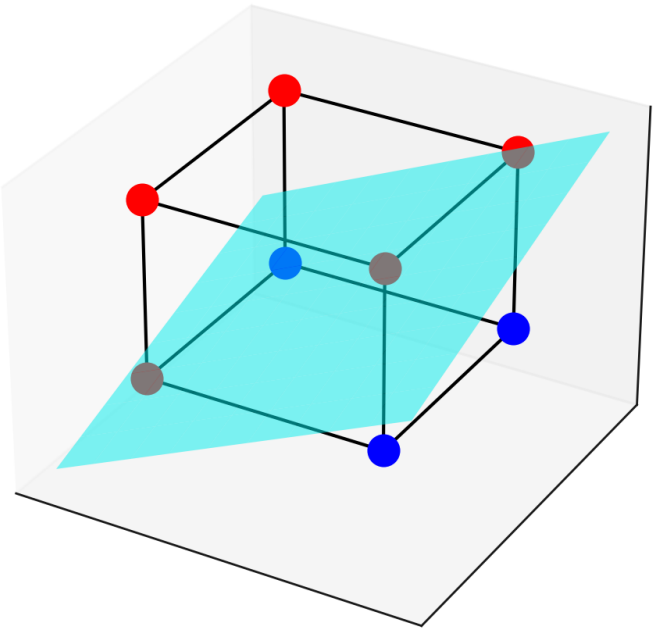

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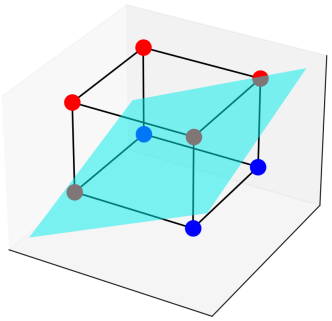

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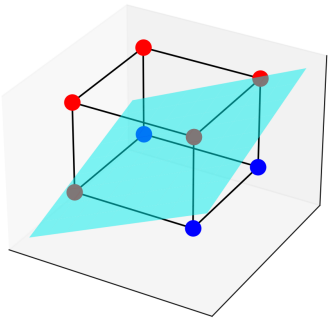

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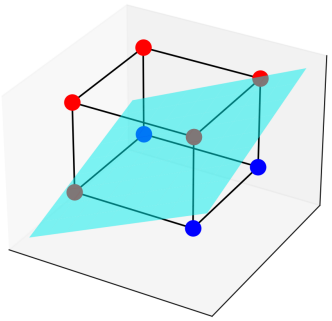

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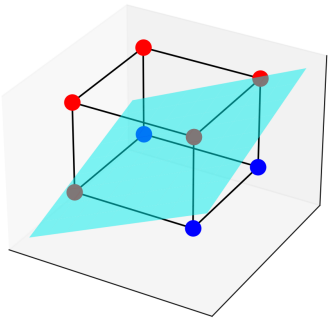

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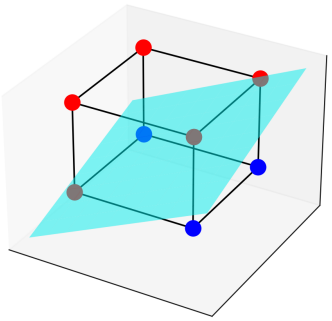

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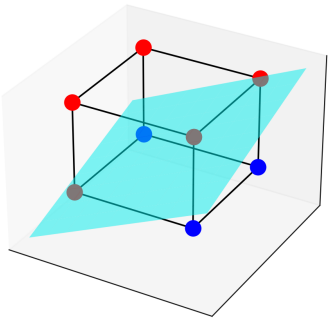

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- For any given input $(a_1, \ldots, a_n)$ you could make 2^n different SSR instances (by varying $\Delta = (\delta_1, \ldots \delta_n)$). If you can solve SSR for a specific set of $\Delta_1, \ldots, \Delta_n \in \{\pm 1\}^n$ (those that correspond to basic feasible solution) you can solve SSR efficiently for all the other instances!

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- Works for linear combinations of arbitrary (but fixed) set of real (even transcendental) numbers, for eg. $\sum_{i=1}^{n} \delta_i (\pi e)^i$ where $\delta_i \in \{-d, \ldots, d\}$

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USSR

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Output: $\operatorname{sgn}\left(\sum_{i=1}^{n} \delta_i \sqrt{a_i}\right)$

Ok, so what?

- For any given input $(a_1, \ldots, a_n)$ you could make 2^n different SSR instances (by varying $\Delta = (\delta_1, \ldots \delta_n)$). If you can solve SSR for a specific set of $\Delta_1, \ldots, \Delta_n \in \{\pm 1\}^n$ (those that correspond to basic feasible solution) you can solve SSR efficiently for all the other instances!

USSR

Input: $1 \leq a_1, \ldots, a_n \leq n^2$ and $\delta_1, \ldots, \delta_n \in \{\pm 1\}$

Output: $\operatorname{sgn}\left(\sum_{i=1}^{n} \delta_i \sqrt{a_i}\right)$

Theorem

There is a non-uniform polynomial time algorithm for USSR.

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Thank You!