Zeros of P-finite Sequences

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Definition

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of rational numbers is called a P-finite (or holonomic) sequence, if it satisfies

$$a_0(n)u_n = a_1(n)u_{n-1} + a_2(n)u_{n-2} + \cdots + a_d(n)u_{n-d}\,,$$

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Example (Apéry's sequence)

Apéry's proof of irrationality of $\zeta(3)$ relied on showing $u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies

$$n^3 u_n = (34n^3 - 61n^2 + 27n - 5)u_{n-1} - (n-1)^3 u_{n-2}\,.$$

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Theorem (Skolem-Mahler-Lech)

Given a C-finite sequence $u = \langle u_n \rangle_{n=0}^{\infty}$, its set of zeros $\mathcal{Z}(u) = \{n \in \mathbb{N} : u_n = 0\}$ is a union of finitely many arithmetic progressions and a finite set.

Non-degeneracy

Definition

Recall that a C-finite sequence has an exponential-polynomial form

$$u_n = \sum_{i=1}^{s} P_i(n)\lambda_i^n \,,$$

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Any C-finite sequence u may be effectively split into at most $2^{\text{poly}(d)}$ subsequences of the form $\langle u_{Nn+r} \rangle_{n=0}^{\infty}$, which are either identically zero or non-degenerate.

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Skolem-Mahler-Lech

Given a *non-degenerate* C-finite sequence u, its set of zeros $\mathcal{Z}(u) = \{n \in \mathbb{N} : u_n = 0\}$ is finite.

The number of zeros of C-finite sequences

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Theorem (Van der Poorten, Schlickewei 1991)

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Open Problem 1

Close the gap!

What about P-finite sequences?

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Theorem (Bell, Burris, Yeats 2012)

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Open Problem 2 (Special case of Dynamical Mordell–Lang Conjecture)

Is $\mathcal{Z}(u)$ a union of finitely many arithmetic progressions and a finite set for all P-finite sequences u?

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Open Problem 3

Find a non p-adic proof of the Skolem-Mahler-Lech theorem.

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In particular u has $\geq \frac{k}{2}$ zeros. So the number of zeros must depend on both the order d and largest degree k of the coefficients.

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Let u be a P-finite sequence u of order d, satisfying a recurrence with $\deg a_i \leq k$.
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Conjecture

Given a P-finite sequence u of order d, satisfying a recurrence with $\deg a_i \leq k$, its zero set $\mathcal{Z}(u) = \{n \in \mathbb{N} : u_n = 0\}$ is a union of at most $B(d, k)$ arithmetic progressions and single integers.

(That is, the structure of zeros is bounded only by the order of u and the degree of the coefficients.)

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Fix $1 \leq i \leq d$, and denote by $\gamma_n^{(i)}$ the ith component of γ.

Find a (not stupidly big) upper bound $B(d,k)$, such that $\gamma_n^{(i)}$ satisfies a monic P-finite recurrence:

$$\gamma_n^{(i)} = a_1(n)\gamma_{n-1}^{(i)} + \cdots + a_d(n)\gamma_{n-m}^{(i)} \,.$$

of length $m \leq B(d,k)$, with $a_i \in R[x]$.