Extensions of the Todd-Coxeter algorithm

via automata theory

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Finitely presented monoids

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Given a monoid presentation $\langle A \mid R \rangle$, describe the monoid M it defines.

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Find the *largest* monoid M such that

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$$\langle\, a, b \,|\, a^2 = 1, b^4 = 1, ab = b^3 a \,\rangle$$

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The bicyclic monoid (infinite).

$$\left\langle\, a, b, c, d, e \;\middle|\; \begin{matrix} ac = ca, ad = da, bc = cb, bd = db, \\ eca = ce, edb = de, c^2a = c^2ae \end{matrix} \,\right\rangle$$

A monoid with undecidable word problem (Tseytin 1958).

Finitely presented finite monoids

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Given a monoid presentation $\langle A \mid R \rangle$, describe the monoid M it defines, assuming that M is **finite**.

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Naive solution: enumerate all finite monoids, find a presentation for each, check isomorphism via Tietze transformations.

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Table: Number of semigroups of order n, up to isomorphism and anti-isomorphism

n	Semigroups of order n
1	1
2	4
3	18
4	126
5	1160
6	15973
7	836021
8	1843120128
9	52989400714478
10	12418001077381302684
11	?

Cayley graphs

The right Cayley graph $\Gamma_A(M)$ of M wrt a generating set A is an edge labelled graph whose

- vertices are elements of M,
- there is an edge $(x, a, y) \in M \times A \times M$ whenever $x \cdot a = y$.

Todd-Coxeter algorithm

Start with the singleton graph.
Repeat the following kinds of steps in some order:

1. add a new edge and vertex if it is "missing",
2. quotient a pair of vertices to enforce a relation,
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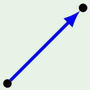

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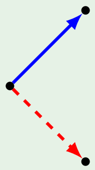

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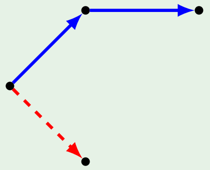

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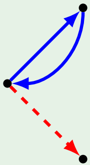

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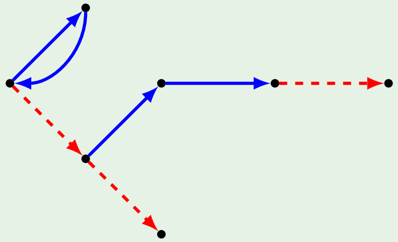

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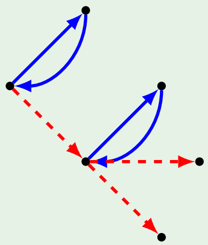

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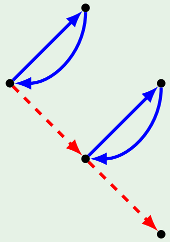

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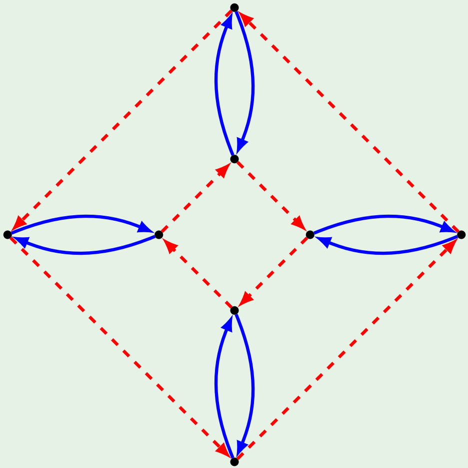

Start with the singleton graph.
Repeat the following kinds of steps in some order:

1. add a new edge and vertex if it is "missing" and "small",
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Theorem (Folklore, c.f. Coleman et al. 2024)

Roughly speaking, provided we always execute the step with the largest number that is applicable, we will eventually produce the right Cayley graph of M, provided M is finite.

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Objects: Graphs with edges labelled by A.
Arrows: Homomorphisms of labelled graphs.

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$$\Gamma_A(M)$$

$$\Gamma_0 \xrightarrow[\psi_1]{} \Gamma_1 \xrightarrow[\psi_2]{} \Gamma_2 \xrightarrow[\psi_3]{} \cdots \longrightarrow \varinjlim \Gamma_n, \qquad \Gamma_0 \xrightarrow{\theta_0} \Gamma_A(M)$$

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$$\theta_0 : \Gamma_0 \longrightarrow \Gamma_A(M), \qquad \theta_1 : \Gamma_1 \longrightarrow \Gamma_A(M)$$

$$\Gamma_0 \xrightarrow[\psi_1]{} \Gamma_1 \xrightarrow[\psi_2]{} \Gamma_2 \xrightarrow[\psi_3]{} \cdots \longrightarrow \varinjlim \Gamma_n$$

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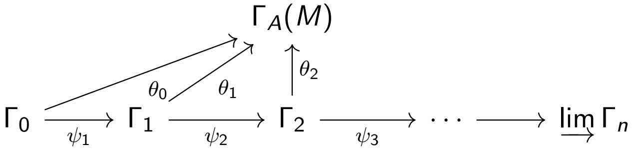

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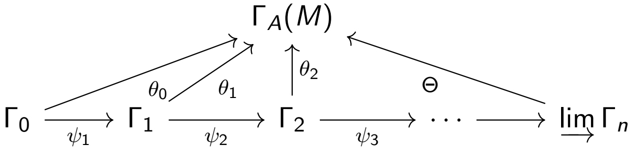

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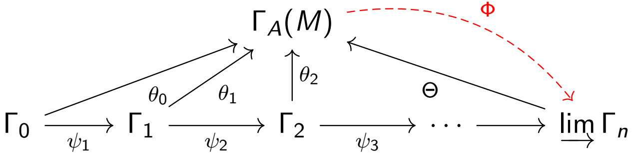

For more details see Stephen 1987 and Coleman et al. 2024.

Applications and extensions

- Enumerate f.p. finite groups and their cosets; (Todd and Coxeter 1936)
- Enumerate f.p. finite semigroups and monoids and congruences on them; (Neumann 1967)
- Enumerate finite $\mathscr{R}$-classes of f.p. inverse monoids; (Stephen 1987)
- Enumerate finite index subgroups of a f.p. group; (Sims 1994)
- Enumerate f.p. finite inverse monoids; (Cutting 2001)
- Enumerate finite index congruences of a f.p. semigroup or monoid; (Anagnostopoulou-Merkouri, C., Mitchell, and Tsalakou 2025)
- And many other, see e.g. (Linton 1995).

Extending to finitely based varieties

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Example

- Variety of commutative monoids $\mathrm{Com} = [xy = yx]$;
- Variety of idempotent monoids (bands) $\mathcal{B} = [x^2 = x]$;
- Variety of semilattices $\mathcal{SL} = [xy = yx, x^2 = x]$.

Problem

Given finitely based variety of monoids $\mathcal{V}$, defined by equations $\mathcal{E}$, and a $\mathcal{V}$-presentation $\mathcal{V}\langle A \mid R \rangle$, describe the monoid M it defines, assuming M is finite.

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Find the *largest* monoid M such that

1. M is generated by the set A,
2. the generators satisfy relations in the set $R \subseteq A^* \times A^*$ and
3. the monoid M satisfies the equations in $\mathcal{E}$.

Motivating example: variety of bands

Theorem (Green and Rees 1952)

Every band M defined by a finite band presentation $\mathcal{B}\langle A \mid R \rangle$ is finite.

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Yet there is no simple way of finding a monoid presentation for M! So we can't use Todd-Coxeter.

n	Order of $F\mathcal{B}(n)$
1	2
2	7
3	160
4	332381
5	2751884514766
6	272622932796281408879065987
n	$\geq 2^{2^{n-1}}$

A related problem

Fix a monoid equation $u = v$, then $\mathrm{Model}(u = v)$ is the decision problem with:
Input: transformations $a_1, \ldots, a_k \in \mathcal{T}_n$
Output: does $\langle a_1, \ldots, a_k \rangle$ satisfy $u = v$ universally?

Theorem (Fleischer and Jack 2020)

$\mathrm{Model}(u = v)$ is in NL. Furthermore, $\mathrm{Model}(x^2 y = x^2)$ is NL-complete.

Example

$$a_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

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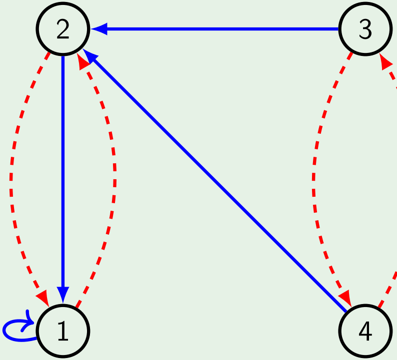

Example ($\mathrm{Model}(xyx = yx)$)

Does $\forall x, y \in M, xyx = yx$ hold in $M = \langle a_1, \ldots, a_k \rangle$?

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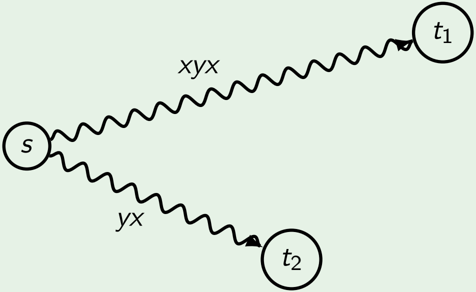

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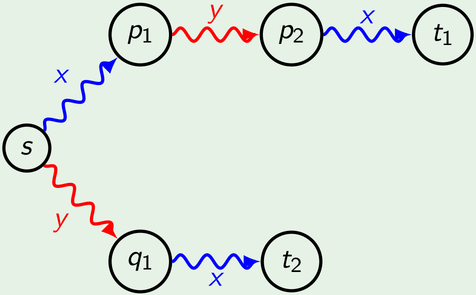

To get complexity, simply guess s, t_1, t_2, p_i, q_i and then guess x and y letter-by-letter.

Todd-Coxeter algorithm in finitely based varieties

Construct the right Cayley graph of M by performing the following kinds of steps:

1. add a new edge and vertex if it is “missing” and “small”,
2. quotient a pair of vertices to enforce a relation,
3. quotient a pair of vertices to enforce a universal equation,
4. quotient a pair of edges to enforce determinism,

Theorem (C. 2026)

Roughly speaking, provided we always execute the step with the largest number that is applicable, we will eventually produce the right Cayley graph of M, provided M is finite.

Further work

- Implement algorithm and check if its any good in practice.
- Is there a good way to reuse universal equation information in-between steps?
- Can something like this work in algebras of distinct signature?
- Generalize to allow equations using both universally and non-universally quantified variables.
- Integrate into framework from Anagnostopoulou-Merkouri et al. 2025 to enumerate finite index congruences.

Thank you for your attention!

References I

Anagnostopoulou-Merkouri, Marina et al. (Dec. 2025). "Computing finite index congruences of finitely presented semigroups and monoids". In: *Mathematics of Computation*. ISSN: 0025-5718. DOI: 10.1090/mcom/4136. URL: http://dx.doi.org/10.1090/mcom/4136.

C., R. (2026). "Automata theoretic methods for computation in semigroups". en. PhD thesis. DOI: 10.17630/STA/1523.

Coleman, T. D. H. et al. (May 2024). "The Todd–Coxeter algorithm for semigroups and monoids". In: *Semigroup Forum* 108.3, pp. 536–593. DOI: 10.1007/s00233-024-10431-z.

Cutting, Andrew (2001). "Todd-Coxeter methods for inverse monoids". PhD thesis. URL: https://hdl.handle.net/10023/15052.

Fleischer, Lukas and Jack, Trevor (2020). "The complexity of properties of transformation semigroups". In: *International Journal of Algebra and Computation* 30.03, pp. 585–606. DOI: 10.1142/S0218196720500125.

References II

- Green, J. A. and Rees, D. (1952). "On semi-groups in which $x^r = x$". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 48.1, pp. 35–40. DOI: 10.1017/S0305004100027341.
- Linton, S. A. (1995). "Generalisations of the Todd-Coxeter Algorithm". In: *Computational Algebra and Number Theory*. Dordrecht: Springer Netherlands, pp. 29–51. DOI: 10.1007/978-94-017-1108-1_3.
- Neumann, B. H. (1967). "Some Remarks on Semigroup Presentations". In: *Canadian Journal of Mathematics* 19, pp. 1018–1026. DOI: 10.4153/CJM-1967-093-2.
- Sims, Charles C. (1994). *Computation with Finitely Presented Groups*. Encyclopedia of Mathematics and its Applications. Cambridge University Press. DOI: 10.1017/CBO9780511574702.
- Stephen, Joseph B. (1987). "Applications of automata theory to presentations of monoids and inverse monoids". PhD thesis.

References III

Todd, J. A. and Coxeter, H. S. M. (1936). "A practical method for enumerating cosets of a finite abstract group". In: *Proceedings of the Edinburgh Mathematical Society* 5.1, pp. 26–34. DOI: 10.1017/S0013091500008221.

Tseytin, G. S. (1958). "An associative calculus with an insoluble equivalence problem". In: *Trudy Mat. Inst. Steklov.* 52, pp. 172–189.