Revisiting Finiteness of Matrix Monoids

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Matrix Monoids + Problem of Interest

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Theorem (Bumpus, Haase, Kiefer, Stoienescu, Tanner 2020)

If $M = \langle A_1, \ldots, A_r \rangle$ *is finite, then every element can be obtained by an exponential length product. (I.e.* $|M| \leq$ *2-exp.)*

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Guess the product leading to a matrix A exceeding the bound (co-NPSPACE=PSPACE) □

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Theorem

If $M = \langle A_1, \ldots, A_r \rangle$ can be conjugated to the integers, then a corresponding matrix P with $PMP^{-1} \subseteq \mathbb{Z}^{n \times n}$ can be found in PTIME.

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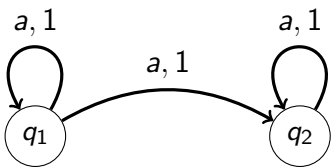

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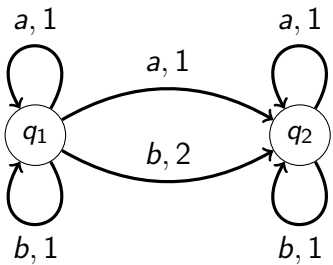

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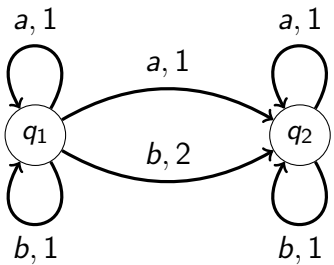

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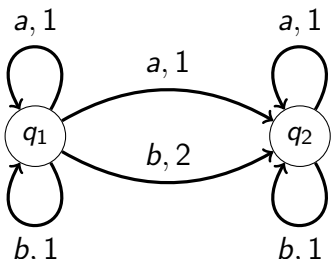

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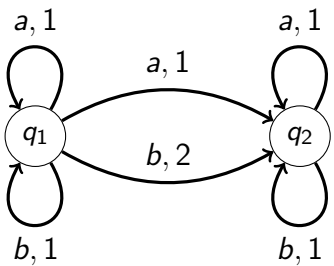

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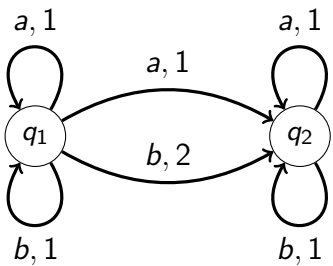

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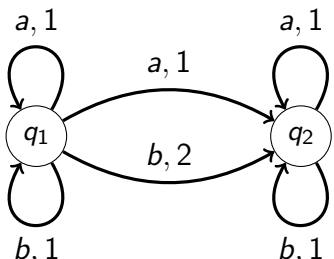

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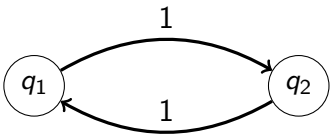

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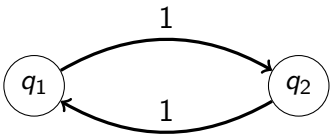

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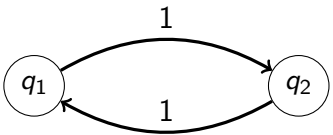

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Idea: M is semi-simple if it is a disjoint union of SCCs.

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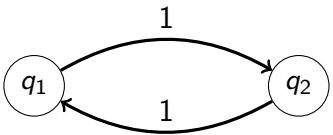

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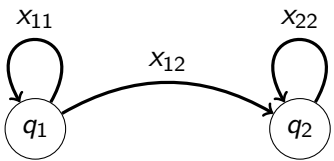

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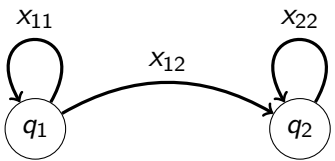

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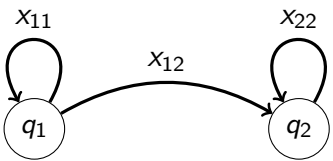

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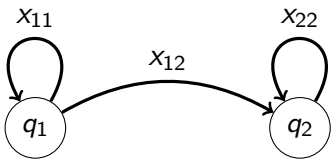

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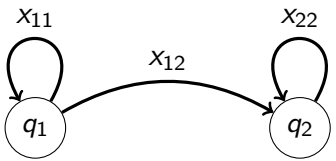

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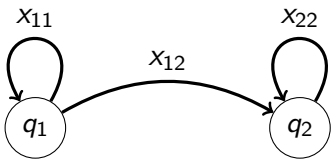
$$\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{N} \}$$

is not semi-simple.



$$\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle = \{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}.$$

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Wedderburn-Malcev Decomposition

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Theorem (This Talk)

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Theorem (Schützenberger 1962)

If $M = \langle A_1, \ldots, A_r \rangle$ is finite and semi-simple, then $|M| \leq (2n+1)^{n^2}$.

Theorem (Bumpus, Haase, Kiefer, Stoienescu, Tanner 2020)

If $M = \langle A_1, \ldots, A_r \rangle$ is finite, then every element can be obtained by an exponential length product. (I.e. $|M| \leq$ 2-exp.)

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To bound entries above the diagonal, use BHKST'20.

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$A = \sum_{\{i_1,\ldots,i_n\} \subseteq \{1,\ldots,2^n\}} (D_1 \ldots D_{i_1-1}) U_{i_1} \ldots U_{i_n} (D_{i_n+1} \ldots D_{2^n})$ □