Shuffle-Raitonal Series, Doubly Rational Series & Shuffle-Finite Series

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Outline

1. A brief of Chen–Fliess Series
2. Shuffle Algebra
3. Shuffle-Rational Series
4. Action of Rational Functions on Formal Series

Notations

- $X = \{x_0, x_1, x_2 \ldots, x_m\}$ be a finite set of noncommuting alphabet.
- X^* is the free monoid generated by X. The elements of X are words, with empty word denoted by $\emptyset$. The subset of nonempty words viz. $X^+ := X^* \setminus \{\emptyset\}$.
- $\mathbb{R}\langle\langle X\rangle\rangle$ is the linear dual to the free algebra generated by X. It is the (Cauchy) algebra of all formal series, where $\mathbb{R}\langle\langle X\rangle\rangle \ni c : X^* \longrightarrow \mathbb{R}$
- A typical way to represent the formal series c is

$$c = \sum_{w \in X^*} c(w)w$$

- $\mathbb{R}\langle X\rangle$: noncommutaive polynomials is the subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ that have finite support.

What is a Chen–Fliess Series ?

Every series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has an associated Chen–Fliess series

$$F_c[u](t) = \sum_{\eta \in X^*} c(\eta)\, F_\eta[u](t), \tag{1}$$

where $E_\emptyset[u] = 1$ and

$$F_{x_i\eta}[u](t) = \int_0^t d\tau\, u_i(\tau) F_\eta[u](\tau)$$

with $x_i \in X$, $\eta \in X^*$, $u \in L_1^m[0, t]$ and $u_0 = 1$ (Fliess, 1981).

Why should anyone care about Chen–Fliess Series?

"Nonlinear dynamical systems" which are modeled by controlled differential equations on $\mathbb{R}^n$ is given by

$$\dot{z} = f(z) + \sum_{i=1}^{m} g(z)u_i(t), \quad z(0) = z_0 \in \mathbb{R}^n$$
$$y(t) = h(z(t))$$

with real analytic vector fields $f, g_1, g_2, \ldots, g_m$ and with a real analytic function h (around z_0),

The local input-output behaviour of such systems viz. a map from $\{u_1, u_2, \ldots, u_m\} \mapsto y$ is modeled by a Chen–Fliess series. (Fliess, 1981)

A small goodie bag!

One can ask for the converse:

What conditions on the formal power series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ are necessary and sufficient that the coressponding Chen–Fliess series has a canonical realization as a local input-to-ouput map of a controlled nonlinear system ?

Fliess & Reutenauer (1982) answered this question:

- The coefficients of c has a Cauchy Growth
- Lie rank of c is finite,.

Shuffle Product

The shuffle product of two words is inductively defined as

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $\eta, \xi \in X^*$, $x_i, x_j \in X$, and $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$.

Example

Examples

$$x_1 \sqcup\!\sqcup x_2 = x_1x_2 + x_2x_1$$
$$x_1x_2 \sqcup\!\sqcup x_1 = 2x_1^2x_2 + x_1x_2x_1$$
$$x_1 \sqcup\!\sqcup x_1 = 2x_1^2$$

Observe that the shuffle product is commutative by defintion.

The shuffle product is then extended bilinearly to series in $\mathbb{R}\langle\langle X\rangle\rangle$. Say for $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$

$$(c \sqcup\!\sqcup\, d)(\eta) = \sum_{\substack{u,v\in X^* \\ \eta\in\mathrm{supp}(u \sqcup\!\sqcup\, v)}} c(u)d(v)$$

$(\mathbb{R}\langle\langle X\rangle\rangle, \sqcup\!\sqcup\,)$ is an associative and commutaive algebra.

For $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$, the product of their corresponding Chen–Fliess series is a Chen–Fliess series given by the shuffle proudct of c and d (Ree, 1958)

$$F_c.F_d[u] = F_{c \sqcup\!\sqcup\, d}[u]$$

Shuffle Group

The subset of non proper series (constant termis nonzero) in $\mathbb{R}\langle\langle X\rangle\rangle$ constitutes a group under the shuffle product.

The shuffle inverse of a non proper series is taken to be

$$c^{\sqcup\!\sqcup\, -1} = (c(\emptyset)(1 - c'))^{\sqcup\!\sqcup\, -1} = c(\emptyset)^{-1}(c')^{\sqcup\!\sqcup\, *}, \tag{2}$$

where $c' := 1 - c/c(\emptyset)$ is proper, and $(c')^{\sqcup\!\sqcup\, *} = \sum\limits_{k\in\mathbb{N}_0} (c')^{\sqcup\!\sqcup\, k}$. Here $(c')^{\sqcup\!\sqcup\, k} \triangleq c' \sqcup\!\sqcup\, (c')^{\sqcup\!\sqcup\, k-1}$ with $(c')^{\sqcup\!\sqcup\, 0} = 1\emptyset$.

Example

Let $c = 1 - x_1 \ \in \mathbb{R}\langle\langle X\rangle\rangle$ so that $c' = x_1$. Then $c^{\sqcup\!\sqcup\, -1} = x_1^{\sqcup\!\sqcup\, *} = \sum_{k\in\mathbb{N}_0} k!\, x_1^k$.

Rational Closures & Rational Series

Definition

An $\mathbb{R}$-subalgebra $\mathscr{F}$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ is said to be *rationally closed* if and only if the inverse of all invertible elements of $\mathscr{F}$ belongs to $\mathscr{F}$.

The *rational closure* of an $\mathbb{R}$-subalgebra $\mathscr{F}'$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra $\mathscr{F}$ containing $\mathscr{F}'$.

Classically, *rational series* are defined to be those in the rational closure of the $\mathbb{R}$-subalgebra of polynomials $\mathbb{R}\langle X\rangle$, where the $\mathbb{R}$-algebra structure on $\mathbb{R}\langle\langle X\rangle\rangle$ is under Cauchy product (Berstel & Reutenauer).

Rational Series

The following statements are known to be equivalent:

- Series c is rational.
- Series c is recognizable (there exists a linear representation λ, μ, γ) where $\lambda, \gamma \in \mathbb{R}^n$ and $\mu : X^* \longrightarrow M_n(\mathbb{R})$ is a monoid morphism and $\forall\, w \in X^*$

  $$c(w) = \lambda^t \mu(w) \gamma$$

  (Schutzenburger Theorem) (Schützenberger, 1961).
- Series c has a Hankel matrix with finite rank n (Fliess, 1974).
- Operator $y = F_c[u]$ has a canonical bilinear state space realization (Fliess, 1981)

$$\dot{z} = (\mu(x_0) + \sum_{i=1}^{m} \mu(x_i) u_i(t)) \quad z(0) = \gamma$$

$$y = \lambda^t z$$

All is well, but where is the question?

- What is the rational closure of subalgebra of polynomials in the shuffle algebra ?
- Can one give “decent” notion of recognizability ?
- What is the correpsonding canonical realization as a nonlinear system ?
- What are the other possible equivalent notions ?
- What is the machine that recognizes rational closure of polynomials in shuffle algebra ?

Shuffle-Rational

Definition

A series c is said to be shuffle rational if it is in the rational closure of $\mathbb{R}\langle X\rangle$ in $(\mathbb{R}\langle\langle X\rangle\rangle, ⧢)$.

Equivalently, c can be written as finite number of shuffle products of polynomials shuffled with finite number of shuffle inverse of polynomials

$$c = (c_1 ⧢ c_2 ⧢ \cdots c_k) ⧢ (d_1 ⧢ d_2 ⧢ \cdots d_m)^{⧢ -1}$$

where $\{c_i, d_j\}_{\substack{i=1:k \\ j=1:m}} \in \mathbb{R}\langle X\rangle$.

Rational Functions on Formal Series

Let $p, q \in \mathbb{R}[y]$ and $c \in \mathbb{R}\langle\langle X \rangle\rangle$. Assume $p(y) = \sum_{i=0}^{k} a_i y^i$, where $k \in \mathbb{N}_0$:
The composition of p and c is defined as

$$p(c) = \sum_{i=0}^{k} a_i c^{\sqcup\!\sqcup\, i}.$$

Extending the definition to rational functions $\frac{p}{q} \in \mathbb{R}(y)$gives

$$\frac{p}{q}(c) = p(c) \sqcup\!\sqcup\, q(c)^{\sqcup\!\sqcup\, -1}.$$

provided $c(\emptyset)$ is not root of $q(y)$

These definitions can be generalized to action of mutivariate polynomials on k-tuples of series as the shuffle product is $\mathbb{R}$-bilinear. Say $p = y_1 y_2 \in \mathbb{R}[y_1, y_2, \ldots, y_k]$

$$y_1 y_2(c_1, c_2, \ldots, c_k) = c_1 \sqcup\!\sqcup\, c_2$$

where and $c_i \in \mathbb{R}\langle\langle X \rangle\rangle$, $i = 1, 2, \ldots k$.

In general, given a rational function $\frac{p}{q} \in \mathbb{R}(y_1, y_2, \ldots, y_n)$ and formal series $c_1, c_2, \ldots, c_k \in \mathbb{R}\langle\langle X \rangle\rangle$, then

$$\frac{p}{q}(c_1, c_2, \ldots, c_k) = p(c_1, c_2, \ldots, c_k) \sqcup\!\sqcup\, q(c_1, c_2, \ldots, c_k)^{\sqcup\!\sqcup\, -1}$$

provide $(c_1(\emptyset), c_2(\emptyset), \ldots, c_k(\emptyset))$ is not a root of the denominator polynomial q.

Definition (Shuffle-Recognizable)

A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is said to be *shuffle-recognizable* if $c = p/q\left(\sum_{w \in X^*} \lambda^T \mu(w) \gamma \, w \right)$, where $\lambda^T = (\lambda_1^T \times \lambda_2^T \times \cdots \times \lambda_k^T)$, $\mu = (\mu_1 \times \mu_2 \times \cdots \times \mu_k)$, and $\gamma = (\gamma_1 \times \gamma_2 \times \cdots \times \gamma_k)$. Here $\lambda_i, \gamma_i \in \mathbb{R}^{n_i}$ and $\mu_i : X^* \longrightarrow N_n(\mathbb{R})$ is a monoid morphism into strictly upper triangular matrices.

The tuple $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ is called a k^{th}-order *shuffle-representation* of c

Theorem (V. & Gray (2020))

Shuffle Rational = Shuffle Recognizable.

Theorem (V. & Gray (2020))

- *A series c is shuffle-rational*
- *The Chen–Fliess series F_c has a (not necessarily canonical) realization as a nonlinear system*

$$\dot{z} = (BD(\mu_1(x_0), \ldots, \mu_k(x_0)) + \sum_{j=0}^{m} (BD(\mu_1(x_j), \ldots, \mu_k(x_j))u_j(t))$$

$$y = \frac{p}{q}(BD(\lambda_1, \ldots, \lambda_k)^t z)$$

where $z(0) = col(\gamma_1, \ldots, \gamma_k)$

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Thank You

Questions?