On the equivalence between generating functions computed by memory transducers and enumerating functions produced by indexed grammars

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18 June 2026

i
transducer
a^n

$i \mapsto n$

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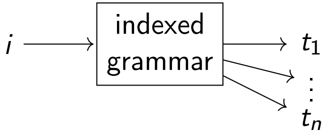
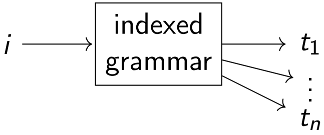


$i \mapsto n$



$i \mapsto n$

$$\mathbb{N} =< \mathbb{N}, \{'' = 0''\}, \{'' - 1''\} >$$

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$$\mathbb{P}(\Gamma, \mathbb{N}) =< (\Gamma \times \mathbb{N})^*, \{" = 0"\} \cup \{\mathcal{P}_\varepsilon\} \cup \{\mathcal{P}_A \mid A \in (\Gamma)\}, \\ \{" - 1"\} \cup \{push(\gamma) \mid \gamma \in \Gamma^*\} >$$

$$" - 1"(A[i] \ldots) := A[" - 1"(i)] \ldots \\ " = 0"(A[\sigma] \ldots) := " = 0"(\sigma)$$

$$\begin{array}{|c|} A[i_1] \\ B[i_2] \\ C[i_3] \\ \vdots \\ \hline \end{array}$$

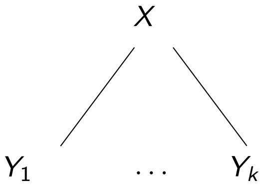
$$\gamma = X_1 X_2 \ldots X_k$$

$$A[i] \ldots \xrightarrow{push(\gamma)} X_1[i] X_2[i] \ldots X_k[i] \ldots$$

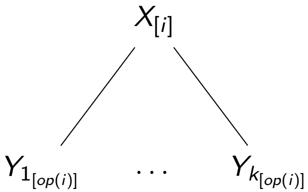
$$q \xrightarrow{\bar{a}} p$$

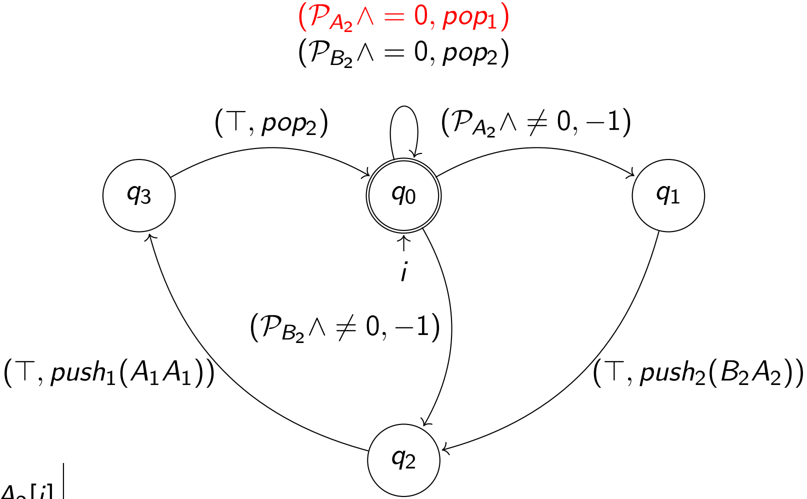
$$q \xrightarrow{g,\bar{a},op} p$$

$$(X \quad) \to (Y_1 ... Y_k \qquad)$$



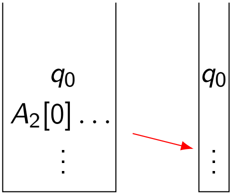
$$(X, g) \to (Y_1 \ldots Y_k, op)$$

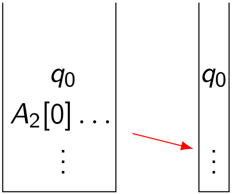




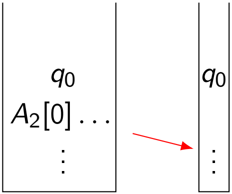
$A_2[i]$

Starting stack


q_0
$A_2[0]\ldots$
$\vdots$
q_0
$\vdots$



$a^{(i+1)!}$ from $q_0, A_1[A_2[i]]$



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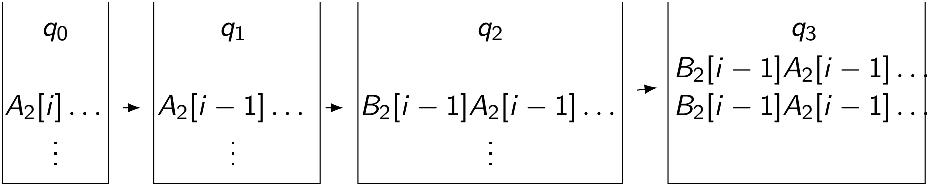

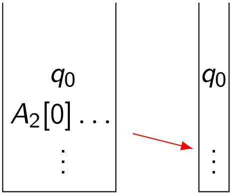

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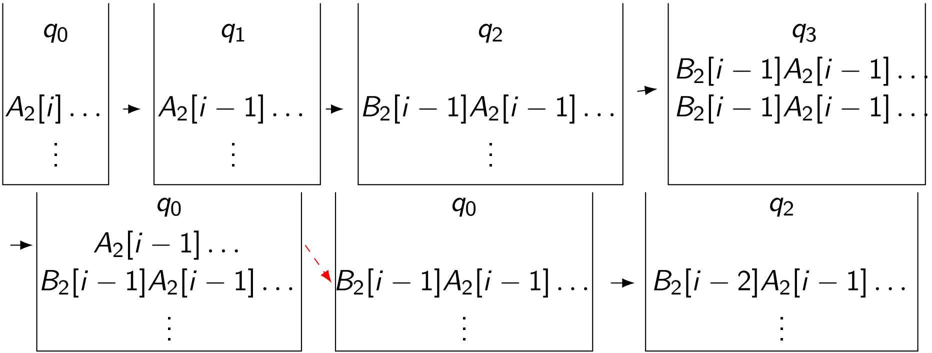

constructive proof

Theorem ($\mathcal{G} \to \mathcal{A}$)

Let $\mathcal{G}$ be a $\mathbb{N}$-indexed grammar, without infinite derivations, producing $u(n)$ derivation trees from the axiom indexed by n. There exists a $\mathbb{P}^2(\mathbb{N})$-transducer that computes $u(n)$.

Theorem ($\mathcal{A} \to \mathcal{G}$)

Let $\mathcal{A}$ be a $\mathbb{P}^2(\mathbb{N})$-transducer, without livelock from any configuration, computing $u(n)$. There exists an $\mathbb{N}$-indexed grammar producing $u(n)$ derivation trees from the axiom indexed by n.

Enumerating trees generated by $\mathbb{N}$-grammars $\longleftrightarrow$ Computed by $\mathbb{P}^2(\mathbb{N})$-transducers

Polynomial recurrent

$u(n) = 2^{2^n}$

D-finite

$u(n) = n!$

Algebraic

Polynomial recurrent : $u_i(n+1) = P_i(f_1(n), ..., f_k(n))$
For all $i \in [0, k]$ and $n \in \mathbb{N}$
With $P_i \in \mathbb{N}[X_1, ..., X_k]$

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APAL 06

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✓

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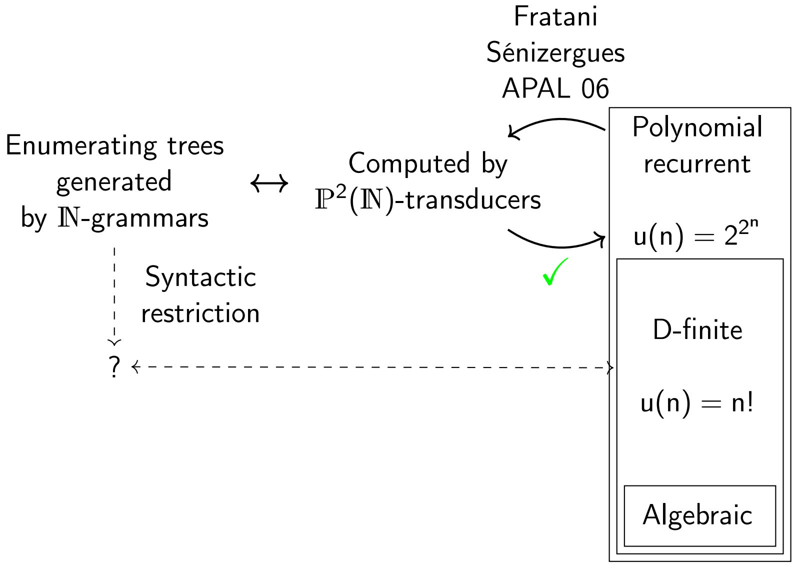

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Theorem ($\mathcal{G} \to \mathcal{A}$)

Let $\mathcal{G}$ be a $\mathbb{N}$-indexed grammar, without infinite derivations, producing $u(n)$ derivation trees from the axiom indexed by n. There exists a $\mathbb{P}^2(\mathbb{N})$-transducer that computes $u(n)$.

Theorem ($\mathcal{A} \to \mathcal{G}$)

Let $\mathcal{A}$ be a $\mathbb{P}^2(\mathbb{N})$-transducer, without livelock from any configuration, computing $u(n)$. There exists an $\mathbb{N}$-indexed grammar producing $u(n)$ derivation trees from the axiom indexed by n.

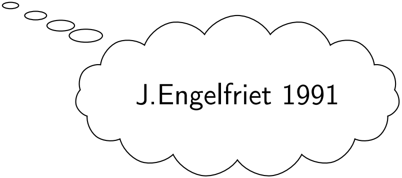
constructive proof

Theorem ($\mathcal{G} \to \mathcal{A}$)

Let $\mathcal{G}$ be a $\mathbb{A}$-indexed grammar, without infinite derivations, producing $u(\sigma)$ derivation trees from the axiom indexed by σ. There exists a $\mathbb{P}^2(\mathbb{A})$-transducer that computes $u(\sigma)$.

Theorem ($\mathcal{A} \to \mathcal{G}$)

Let $\mathcal{A}$ be a $\mathbb{P}^2(\mathbb{A})$-transducer, without livelock from any configuration, computing $u(\sigma)$. There exists an $\mathbb{A}$-indexed grammar producing $u(\sigma)$ derivation trees from the axiom indexed by σ.



$$\mathbb{A} =< \mathcal{S}, \mathcal{P}red, \mathcal{O}p >$$

$$\mathbb{P}(\Gamma, \mathbb{A}) =< (\Gamma \times \mathcal{S})^*, \mathcal{P}red \cup \{\mathcal{P}_\varepsilon\} \cup \{\mathcal{P}_A \mid A \in (\Gamma)\},$$
$$\mathcal{O}p \cup \{push(\gamma) \mid \gamma \in \Gamma^*\} >$$

$$\forall op \in \mathcal{O}p, op(A[\sigma] \dots) := A[op(\sigma)] \dots$$
$$\forall p \in \mathcal{P}red, p(A[\sigma] \dots) := p(\sigma)$$

$$\gamma = X_1 X_2 ... X_k$$
$$A[\sigma] \dots \xrightarrow{push(\gamma)} X_1[\sigma] X_2[\sigma] \dots X_k[\sigma] \dots$$

$$\begin{array}{|c|} A[\sigma_1] \\ B[\sigma_2] \\ C[\sigma_3] \\ \vdots \\ \hline \end{array}$$

Theorem ($\mathcal{G} \to \mathcal{A}$)

Let $\mathcal{G}$ be a $\mathbb{A}$-indexed grammar, without infinite derivations, producing $u(\sigma)$ derivation trees from the axiom indexed by σ. There exists a $\mathbb{P}^2(\mathbb{A})$-transducer that computes $u(\sigma)$.

Theorem ($\mathcal{A} \to \mathcal{G}$)

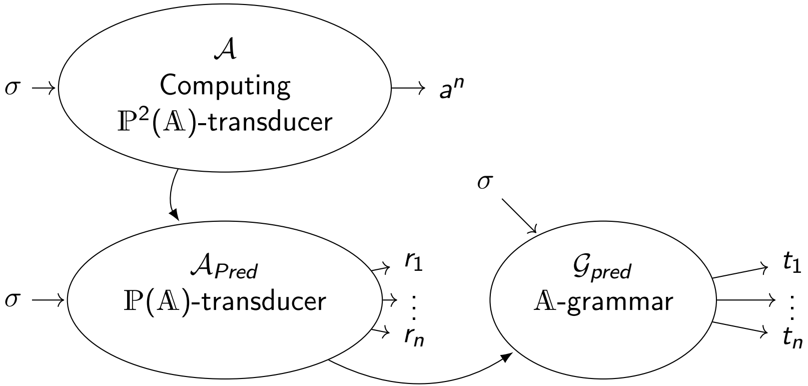
Let $\mathcal{A}$ be a $\mathbb{P}^2(\mathbb{A})$-transducer, without livelock from any configuration, computing $u(\sigma)$. There exists an $\mathbb{A}$-indexed grammar producing $u(\sigma)$ derivation trees from the axiom indexed by σ.

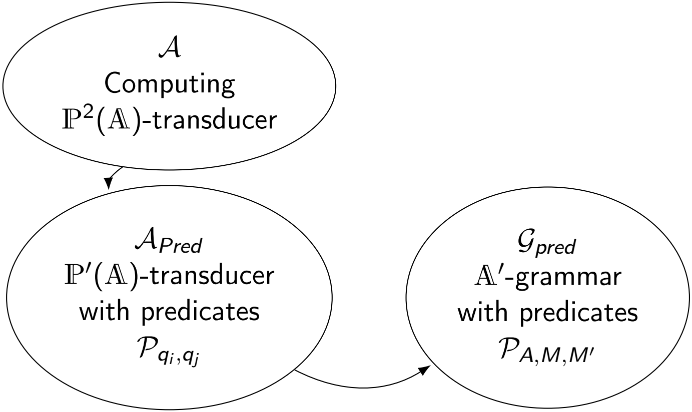
Theorem ($\mathcal{G} \to \mathcal{A}$)

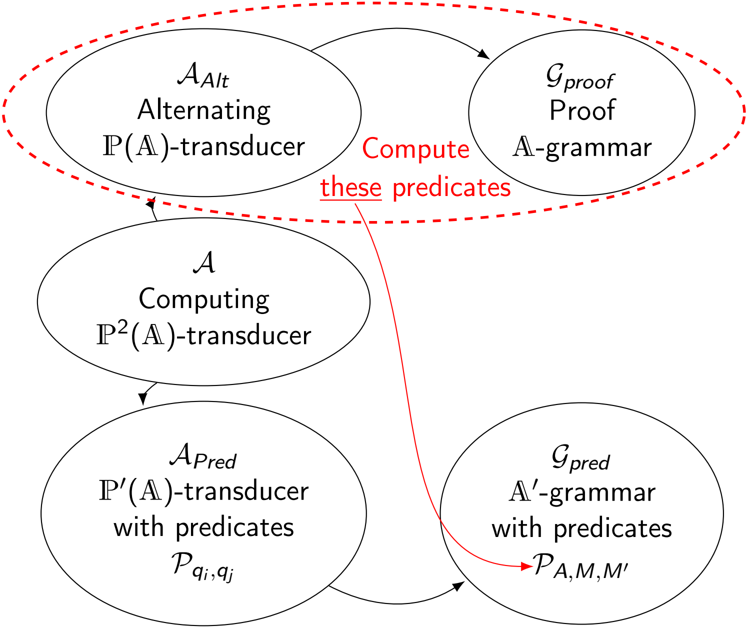
Let $\mathcal{G}$ be a $\mathbb{P}^k(\mathbb{A})$-indexed grammar, without infinite derivations, producing $u(\sigma)$ derivation trees from the axiom indexed by σ. There exists a $\mathbb{P}^{k+2}(\mathbb{A})$-transducer that computes $u(\sigma)$.

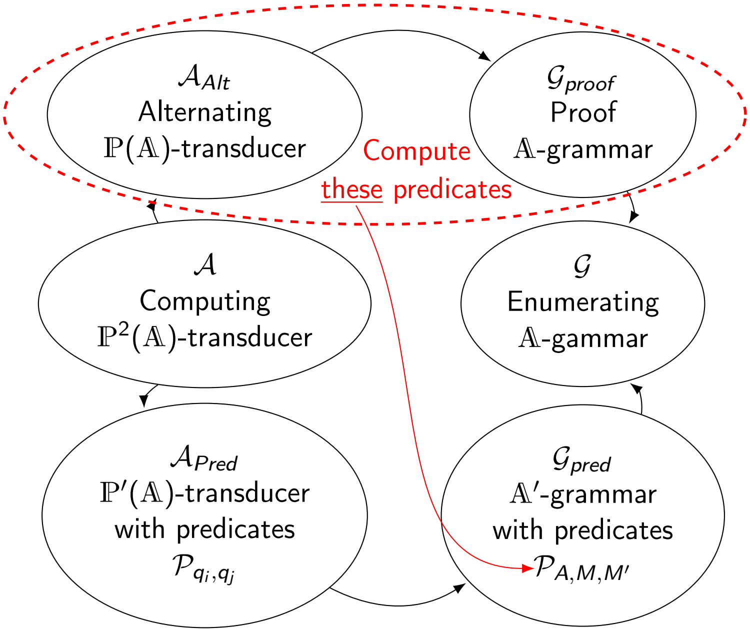
Theorem ($\mathcal{A} \to \mathcal{G}$)

Let $\mathcal{A}$ be a $\mathbb{P}^{k+2}(\mathbb{A})$-transducer, without livelock from any configuration, computing $u(\sigma)$. There exists an $\mathbb{P}^k(\mathbb{A})$-indexed grammar producing $u(\sigma)$ derivation trees from the axiom indexed by σ.


$\mathcal{A}$
Computing
$\mathbb{P}^2(\mathbb{A})$-transducer
σ
a^n
$\mathcal{A}_{Pred}$
$\mathbb{P}(\mathbb{A})$-transducer
σ
r_1
r_n
σ
$\mathcal{G}_{pred}$
$\mathbb{A}$-grammar
t_1
t_n


$\mathcal{A}$
Computing
$\mathbb{P}^2(\mathbb{A})$-transducer
$\mathcal{A}_{Pred}$
$\mathbb{P}'(\mathbb{A})$-transducer
with predicates
$\mathcal{P}_{q_i,q_j}$
$\mathcal{G}_{pred}$
$\mathbb{A}'$-grammar
with predicates
$\mathcal{P}_{A,M,M'}$


$\mathcal{A}_{Alt}$
Alternating
$\mathbb{P}(\mathbb{A})$-transducer
$\mathcal{G}_{proof}$
Proof
$\mathbb{A}$-grammar
Compute
these predicates
$\mathcal{A}$
Computing
$\mathbb{P}^2(\mathbb{A})$-transducer
$\mathcal{A}_{Pred}$
$\mathbb{P}'(\mathbb{A})$-transducer
with predicates
$\mathcal{P}_{q_i,q_j}$
$\mathcal{G}_{pred}$
$\mathbb{A}'$-grammar
with predicates
$\mathcal{P}_{A,M,M'}$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Proof
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$\mathbb{A}$-gammar
$\mathcal{A}_{Pred}$
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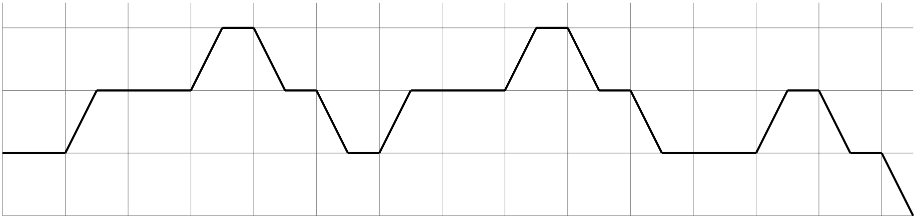
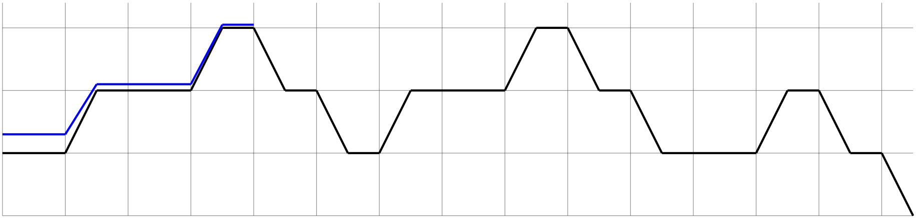
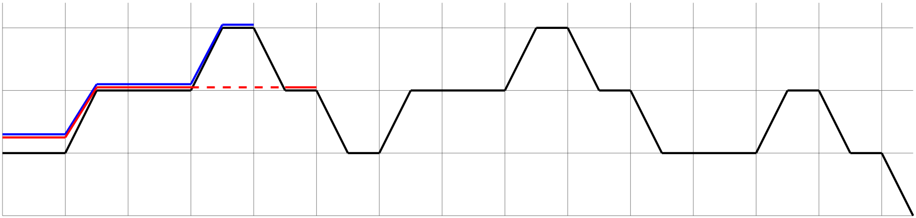
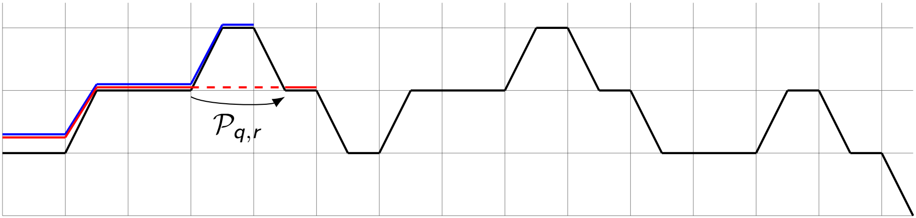
$\mathcal{P}_{q_i,q_j}$
$\mathcal{G}_{pred}$
$\mathbb{A}'$-grammar
with predicates
$\mathcal{P}_{A,M,M'}$

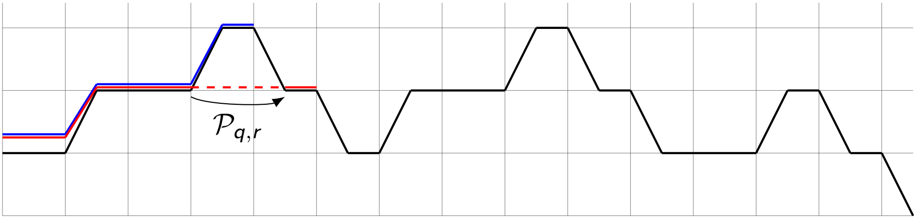




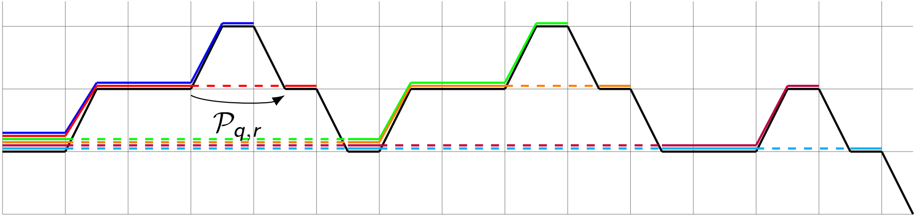



$\mathcal{P}_{q,r}$

$T_2((q, \mathrm{g}) \to (p, \varepsilon, push_1(A_1A_1)))=$ $\{(q, \mathrm{g}) \to (p, \varepsilon, id)\}$



$$T_2((q,\mathrm{g}) \rightarrow (p,\varepsilon, push_1(A_1A_1))) = \begin{array}{l}\{(q,\mathrm{g}) \rightarrow (p,\varepsilon,id)\}\\ \cup\{(q,\mathrm{g}\wedge\mathcal{P}_{p,r}) \rightarrow (r,\varepsilon,id) | r\in\mathcal{Q}\}\end{array}$$



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$$T_2((q, \mathcal{P}_{\varepsilon_2}) \to (p, \varepsilon, pop_1)) = \{\}$$

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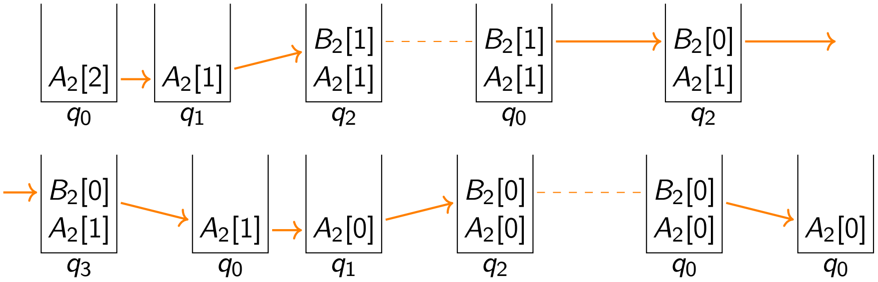
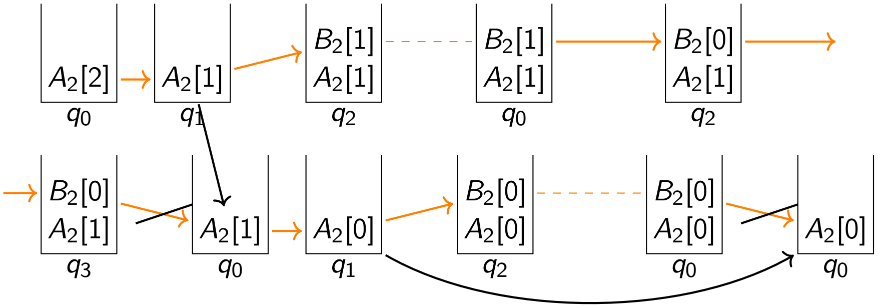
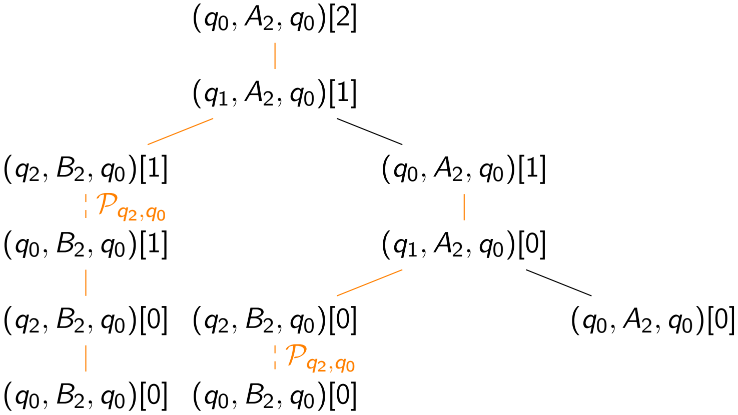

A2[2]
q0
A2[1]
q1
B2[1]
A2[1]
q2
B2[1]
A2[1]
q0
B2[0]
A2[1]
q2
B2[0]
A2[1]
q3
A2[1]
q0
A2[0]
q1
B2[0]
A2[0]
q2
B2[0]
A2[0]
q0
A2[0]
q0


$A_2[2]$
q_0
$A_2[1]$
q_1
$B_2[1]$
$A_2[1]$
q_2
$B_2[1]$
$A_2[1]$
q_0
$B_2[0]$
$A_2[1]$
q_2
$B_2[0]$
$A_2[1]$
q_3
$A_2[1]$
q_0
$A_2[0]$
q_1
$B_2[0]$
$A_2[0]$
q_2
$B_2[0]$
$A_2[0]$
q_0
$A_2[0]$
q_0


$(q_0, A_2, q_0)[2]$
$(q_1, A_2, q_0)[1]$
$(q_2, B_2, q_0)[1]$
$(q_0, A_2, q_0)[1]$
$\mathcal{P}_{q_2,q_0}$
$(q_0, B_2, q_0)[1]$
$(q_1, A_2, q_0)[0]$
$(q_2, B_2, q_0)[0]$
$(q_2, B_2, q_0)[0]$
$(q_0, A_2, q_0)[0]$
$\mathcal{P}_{q_2,q_0}$
$(q_0, B_2, q_0)[0]$
$(q_0, B_2, q_0)[0]$

$$(q_0, A_2, q_0, M^{(1)})[1]$$

$$(q_1, A_2, q_0, M^{(1)})[0]$$

$$(q_2, B_2, q_0, M^{(4)})[0] \qquad (q_0, A_2, q_0, M^{(1)})[0]$$

?

$$(q_0, B_2, q_0, M^{(4)})[0]$$

$$(q_0, A_2, q_0, M^{(1)})[1]$$

$$(q_1, A_2, q_0, M^{(1)})[0]$$

$\mathcal{P}_{A_2, M^{(1)}, M^{(4)}}$

$$(q_2, B_2, q_0, M^{(4)})[0] \qquad (q_0, A_2, q_0, M^{(1)})[0]$$

?

$$(q_0, B_2, q_0, M^{(4)})[0]$$

$(q_0, A_2, q_0, M^{(1)})[1]$

$(q_1, A_2, q_0, M^{(1)})[0]$

$\mathcal{P}_{A_2, M^{(1)}, M^{(4)}}$

$(q_2, B_2, q_0, M^{(4)})[0]$

$(q_0, A_2, q_0, M^{(1)})[0]$

$\mathcal{P}_{A_2, M^{(4)}, M^{(5)}} \wedge (q_3, q_0) \in M^{(5)}$

$(q_0, B_2, q_0, M^{(4)})[0]$

$$(q_0, A_2, q_0, M^{(1)})[1]$$

$$(q_1, A_2, q_0, M^{(1)})[0]$$

$$\mathcal{P}_{A_2, M^{(1)}, M^{(4)}}$$

$$(q_2, B_2, q_0, M^{(4)})[0] \qquad (q_0, A_2, q_0, M^{(1)})[0]$$

$$\mathcal{P}_{A_2, M^{(4)}, M^{(5)}} \wedge (q_3, q_0) \in M^{(5)}$$

$$(q_0, B_2, q_0, M^{(4)})[0]$$

$$\mathcal{P}_{M,M',A_2}(\sigma) \iff$$

$$\overbrace{(q, A_1[A_2[\sigma]stk] \dots) \xrightarrow[\mathcal{A}]{*} \underbrace{(r, A_1[A_2[\sigma]stk] \dots) \xrightarrow[\mathcal{A}]{*} (p, A_1[stk] \dots)}_{\exists (r,p) \in M}}^{\forall (q,p) \in M'}$$