15/06/2026

SAMSA 2026

# Is the linear Zariski closure of a context free language of matrices computable?


Yahia Idriss BENALIOUA

University of Warsaw

# Zariski topologies (on $\mathbb{Q}^d$)

**Def:** Zariski topology

closed sets: common zeroes of polynomials

$P_1, \dots, P_h \in \mathbb{Q}[x_1, \dots, x_d]$

**Def:** Linear Zariski topology

closed sets: Finite unions of vector subspaces of $\mathbb{Q}^d$

# Zariski topologies (on $\mathbb{Q}^d$)

**Def:** Zariski topology

closed sets: common zeroes of polynomials $P_1, \dots, P_n \in \mathbb{Q}[x_1, \dots, x_d]$

**Def:** Linear Zariski topology

closed sets: Finite unions of vector subspaces of $\mathbb{Q}^d$

**Def:** $X$ irreducible

$\forall X_1, X_2$ closed

$X = X_1 \cup X_2 \Rightarrow X = X_1$ or $X = X_2$

# Zariski topologies (on $\mathbb{Q}^d$)

**Def:** Zariski topology

closed sets: common zeroes of polynomials $P_1, \dots, P_h \in \mathbb{Q}[x_1, \dots, x_d]$

**Def:** Linear Zariski topology

closed sets: Finite unions of vector subspaces of $\mathbb{Q}^d$

**Def:** $X$ irreducible

$\forall X_1, X_2$ closed

$X = X_1 \cup X_2 \Rightarrow X = X_1$ or $X = X_2$

**Prop:** $\forall X$ closed

$\exists X_1, \dots, X_h$ irreducible s.t.

$X = X_1 \cup \dots \cup X_h$

# Zariski topologies (on $\mathbb{Q}^d$)

**Def:** Zariski topology

closed sets: common zeroes of polynomials $P_1, \dots, P_n \in \mathbb{Q}[x_1, \dots, x_d]$

**Def:** Linear Zariski topology

closed sets: Finite unions of vector subspaces of $\mathbb{Q}^d$

**Def:** $X$ irreducible

$\forall X_1, X_2$ closed

$X = X_1 \cup X_2 \Rightarrow X = X_1$ or $X = X_2$

**Prop:** $\forall X$ closed

$\exists X_1, \dots, X_n$ irreducible s.t.

$$X = \underset{\text{vector spaces}}{X_1 \cup \cdots \cup X_n}$$

linear case:

$\dim(X) := \max_i (\dim(X_i))$

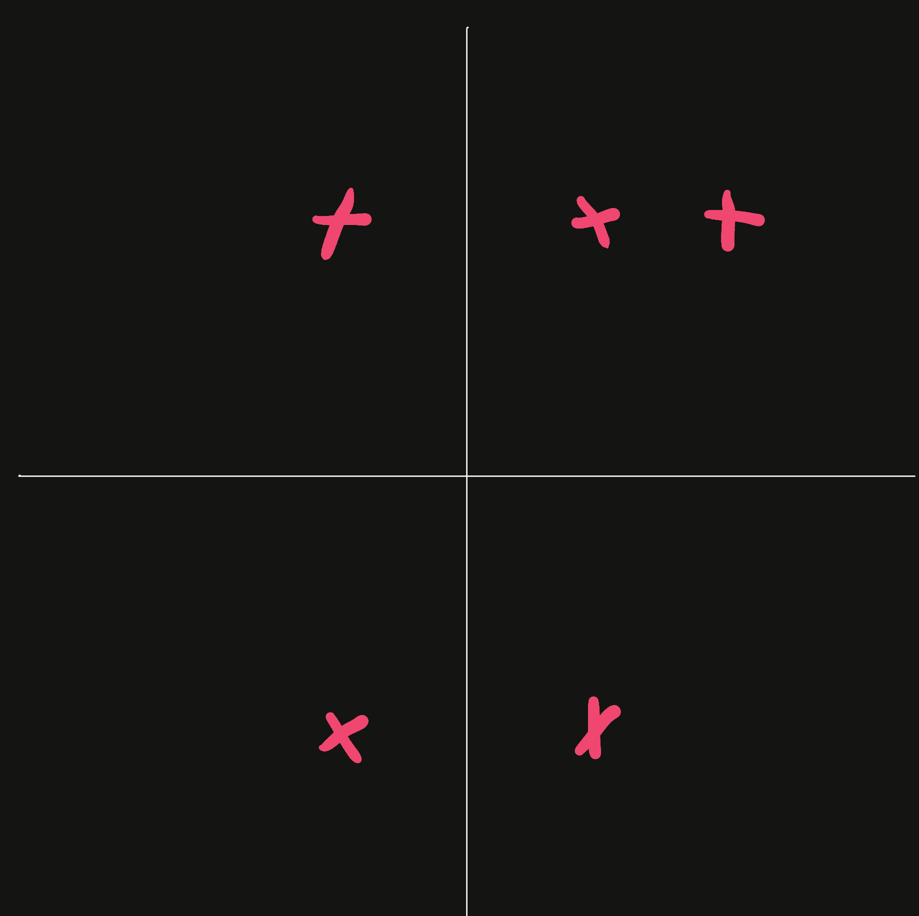
$\text{length}(X) := n$ (number of components)

# Closures

**Not.** $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology

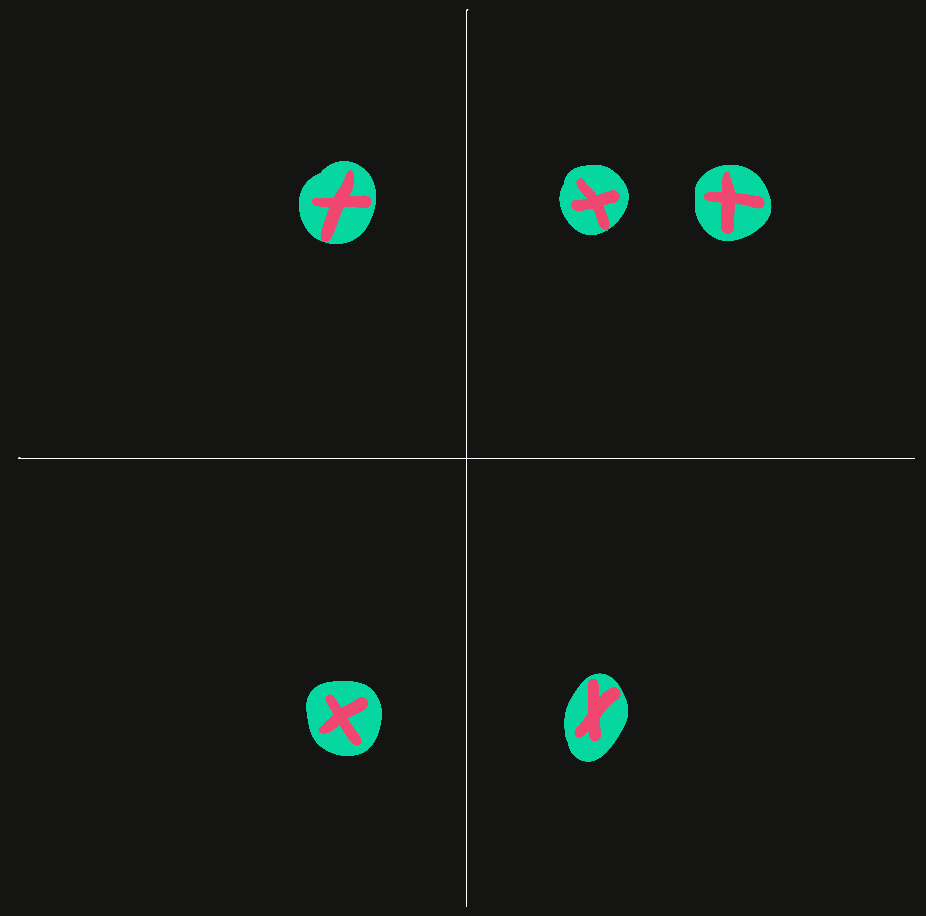


# Closures

Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topotogy

$\overline{X}^\ell$ : Linear Zariski topotogy

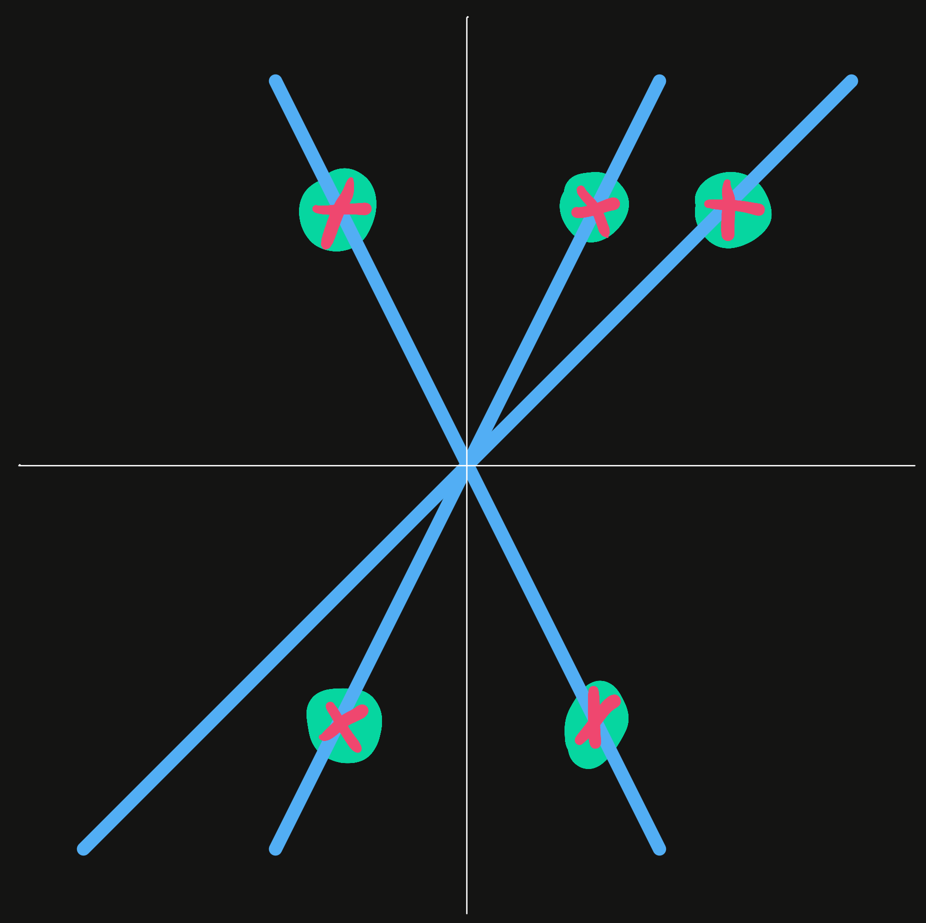


# Closures

**Not.** $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology

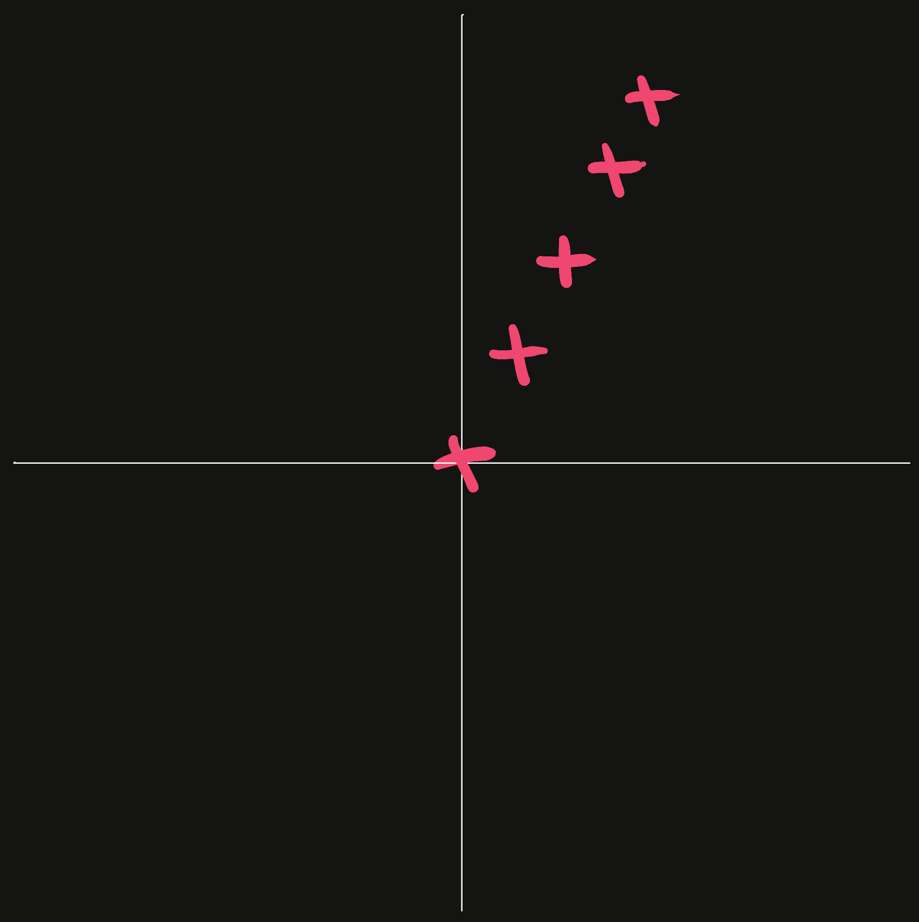


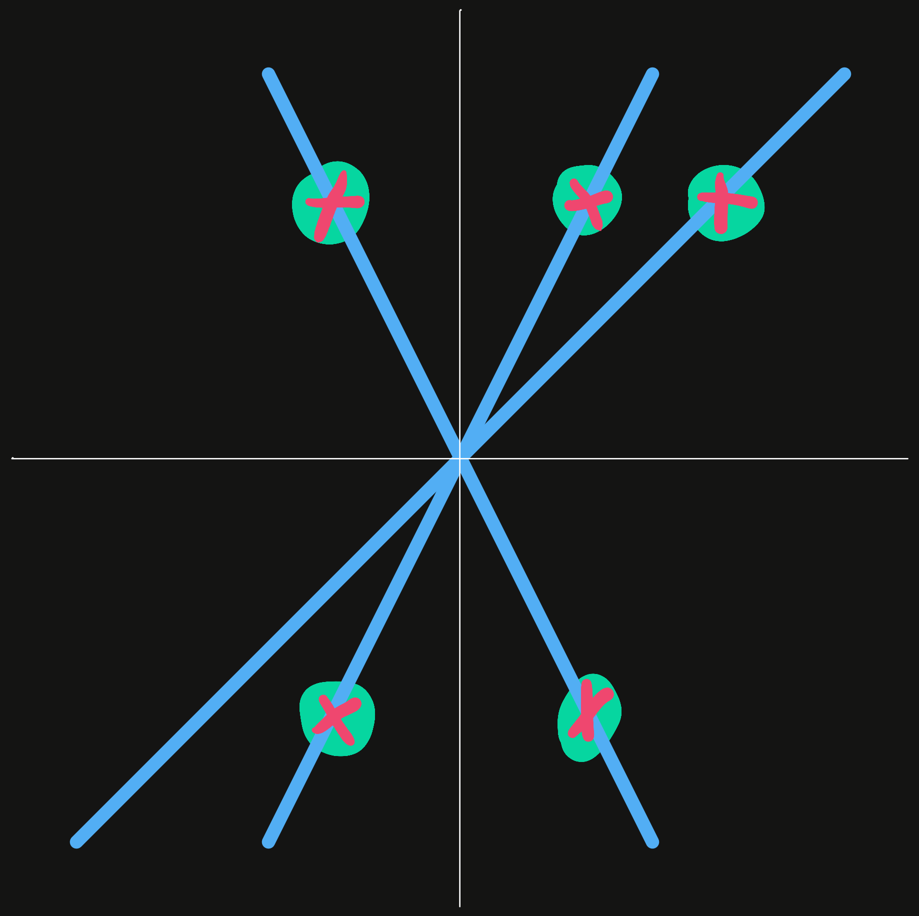
# Closures

Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology



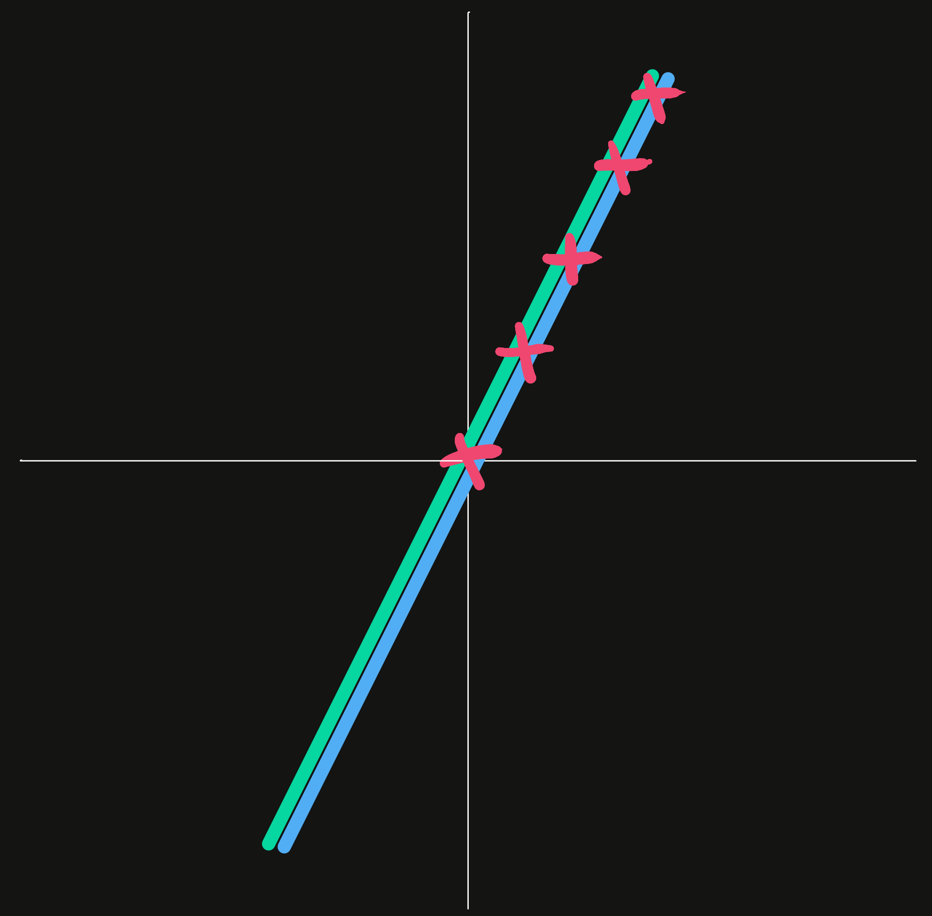


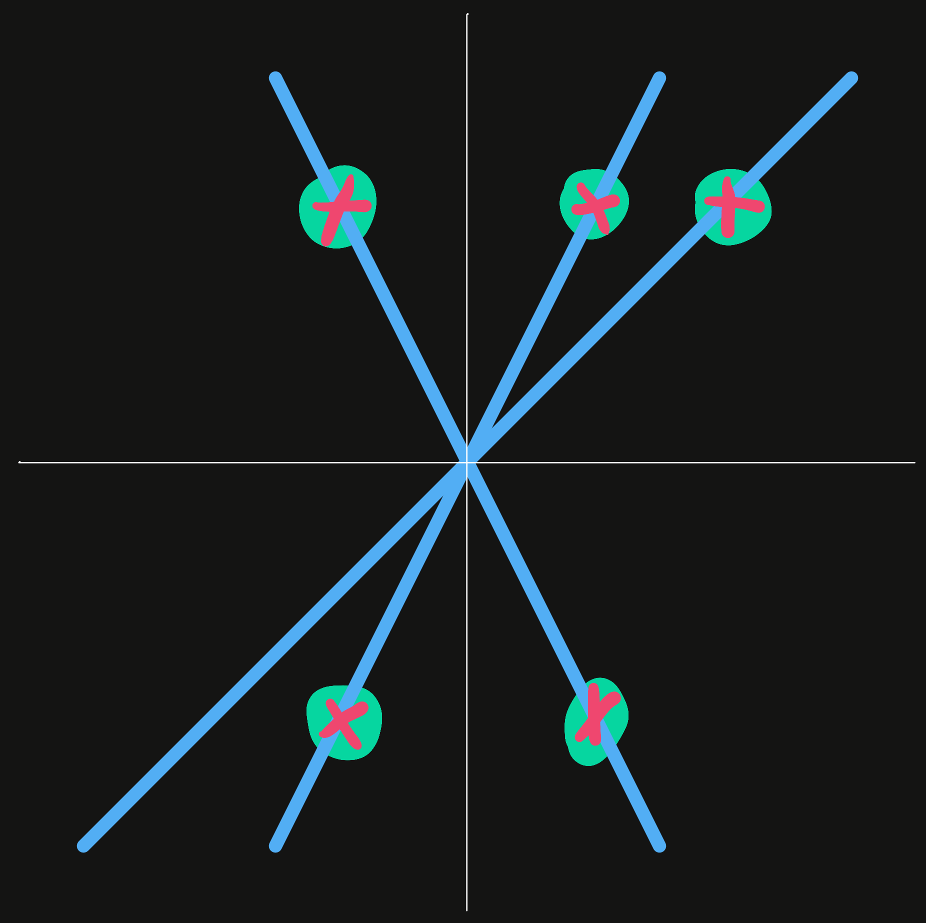
# Closures

Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology



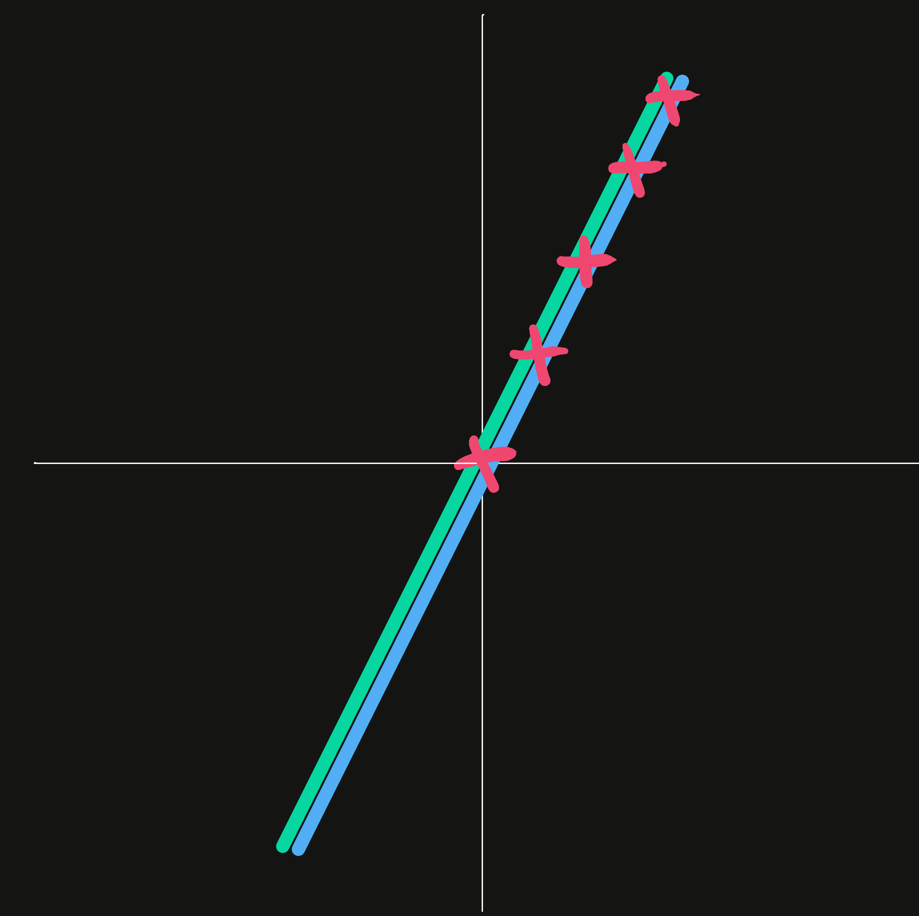


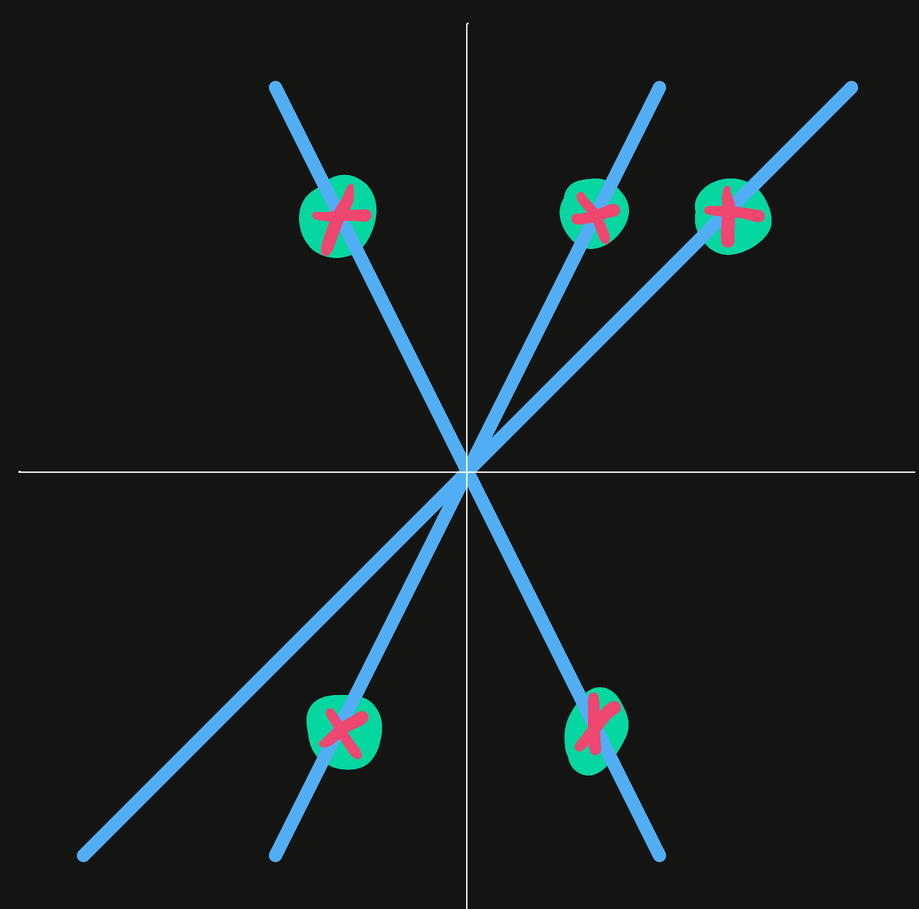
# Closures

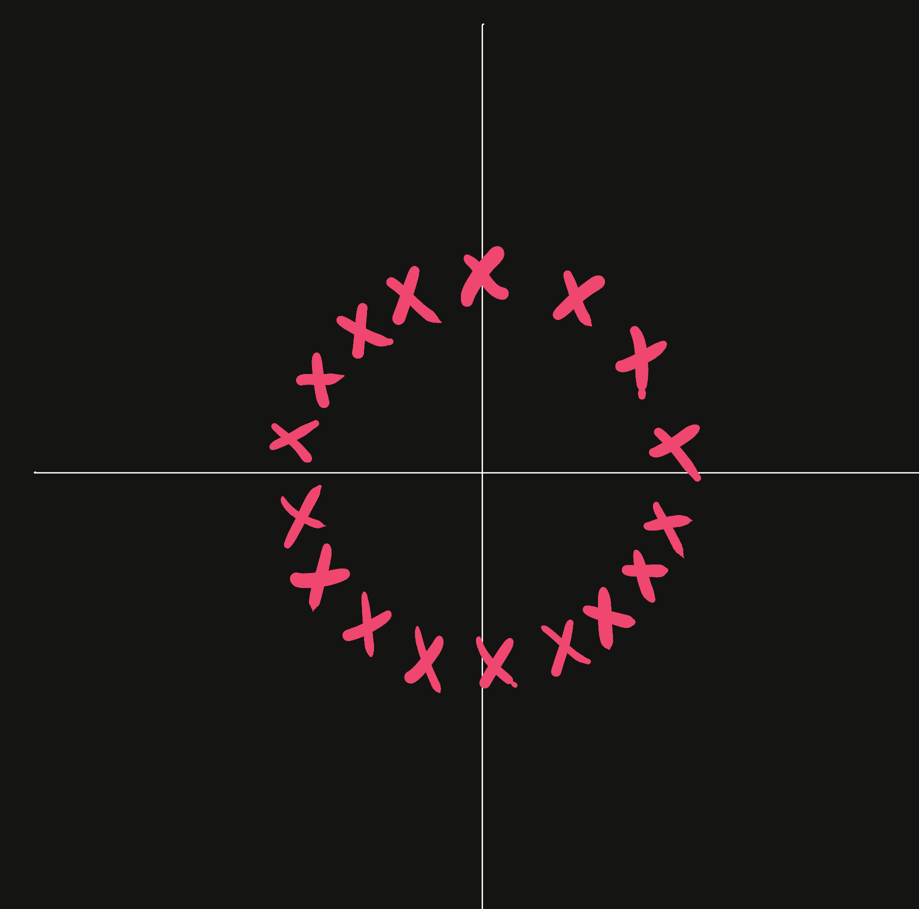
Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology





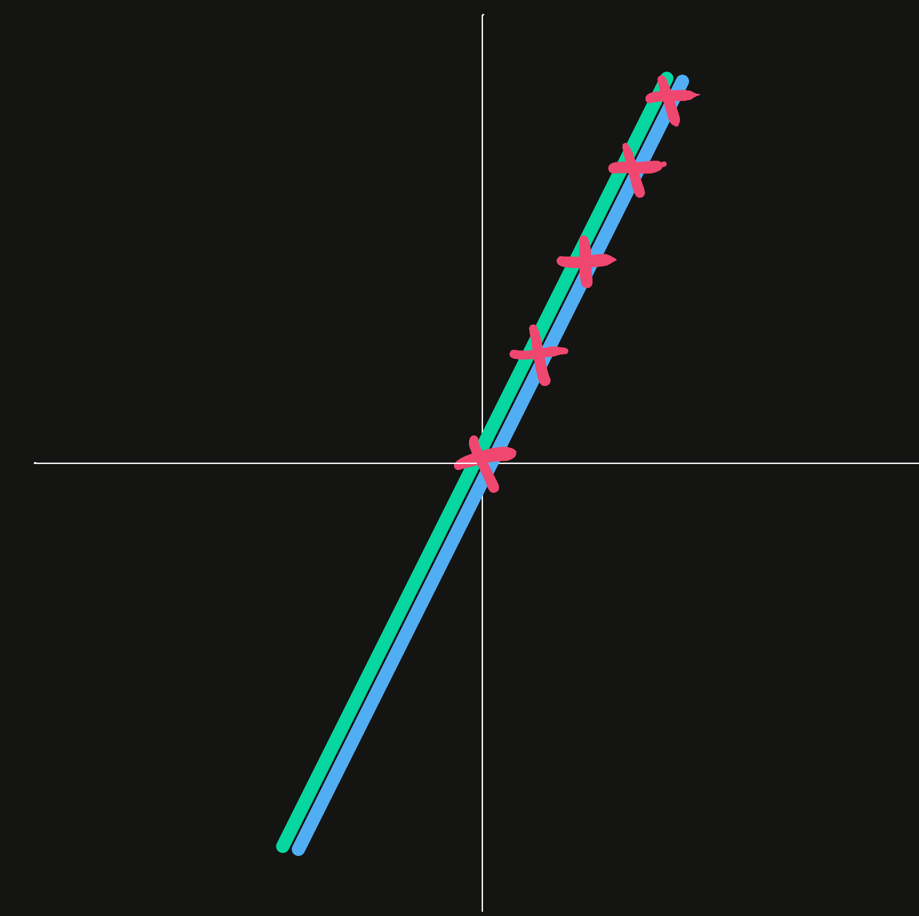


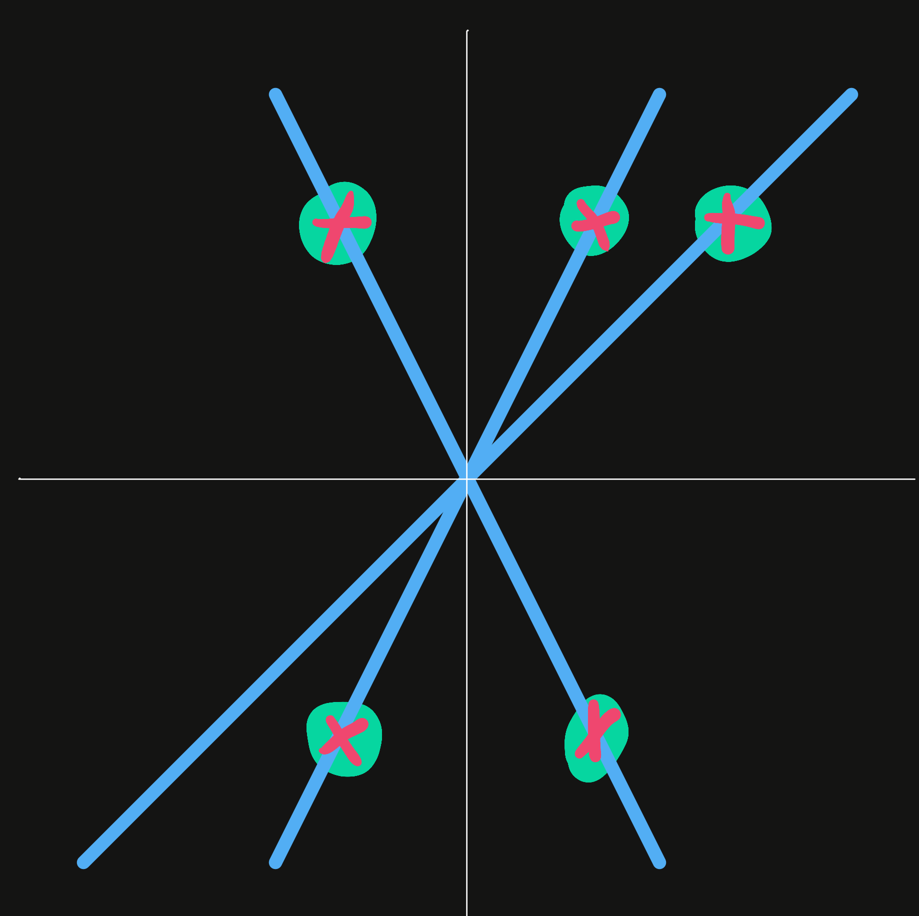
# Closures

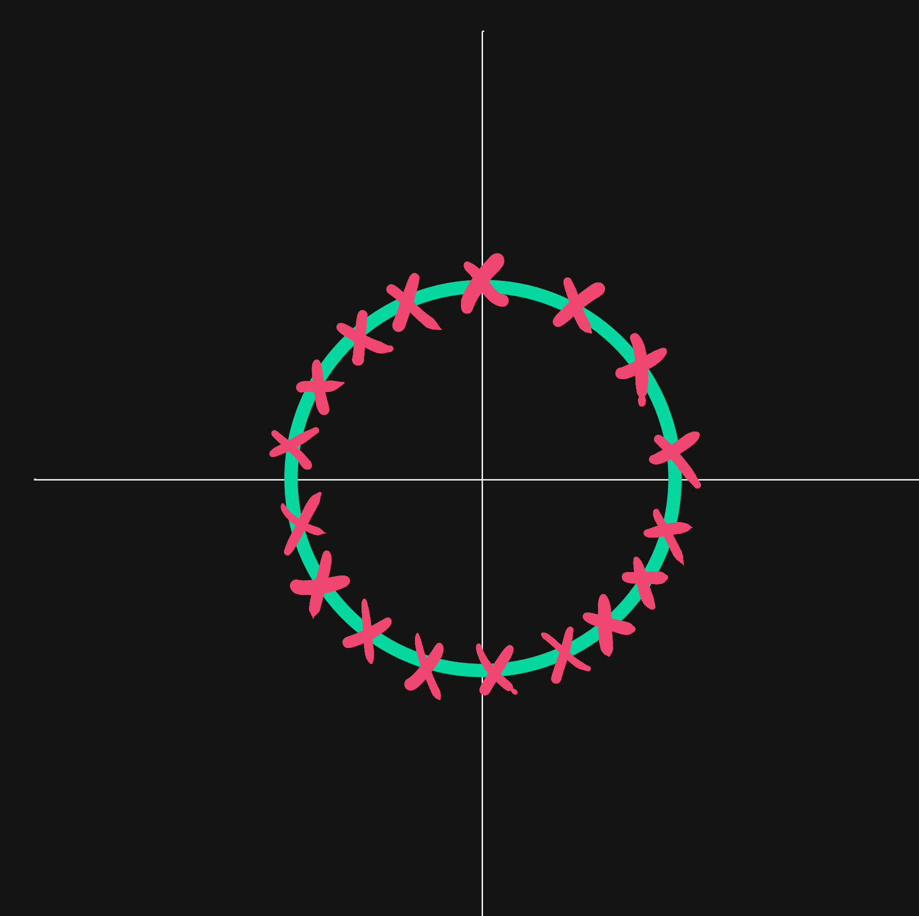
Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^{z}$ : Zariski topology

$\overline{X}^{\ell}$ : Linear Zariski topology





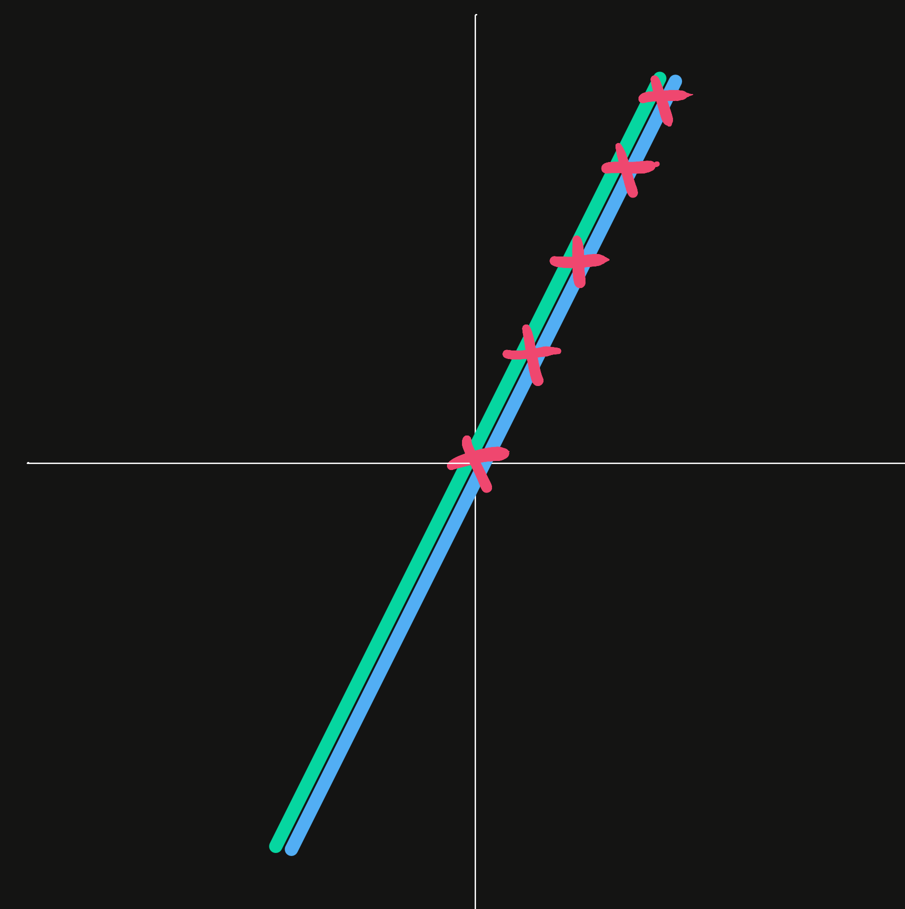


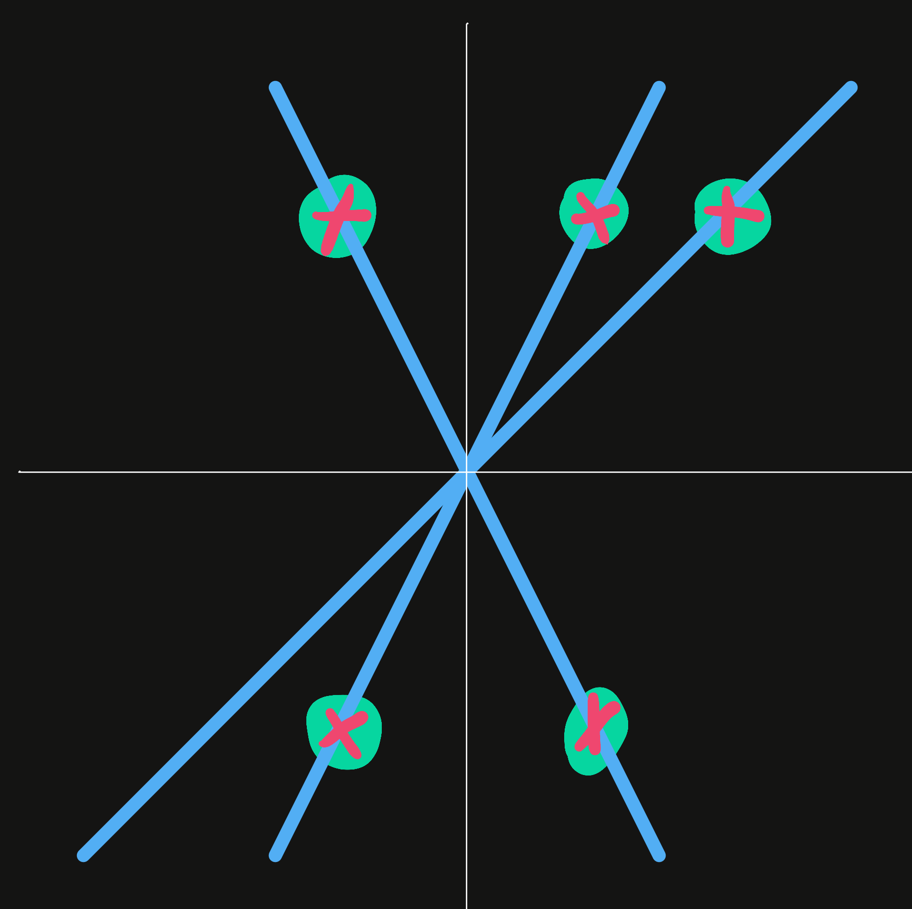
# Closures

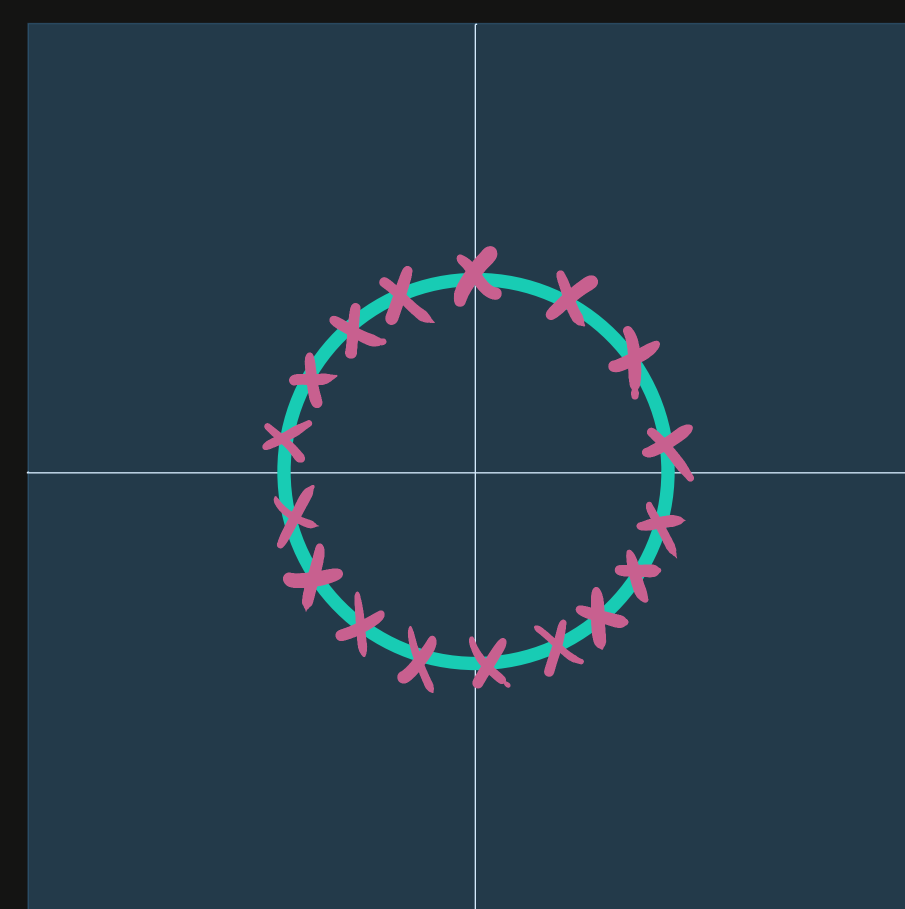
Not. $X \subseteq \mathbb{Q}^d$ closure of $X$ in

$\overline{X}^z$ : Zariski topology

$\overline{X}^\ell$ : Linear Zariski topology







# The problem

$\Sigma$ finite alphabet

Identify $\mathbb{Q}^{d\times d}$ with $\mathbb{Q}^{d^2}$

**Def:** Closure problem

In: $L \subseteq \Sigma^*$

$\mu : \Sigma^* \longrightarrow \mathbb{Q}^{d\times d}$ monoid morphism

Out: $\overline{\mu(L)}^{z}$

# The problem

$\Sigma$ finite alphabet

Identify $\mathbb{Q}^{d\times d}$ with $\mathbb{Q}^{d^2}$

**Def:** Lin Closure problem

**In:** $L \subseteq \Sigma^*$

$\mu : \Sigma^* \to \mathbb{Q}^{d\times d}$ monoid morphism

**Out:** $\overline{\mu(L)}^{\ell}$

**Prop:** Lin closure $\leqslant$ closure

**Proof:** $\overline{X}^{z} = X_1 \cup \cdots \cup X_n \Rightarrow \overline{X}^{\ell} = \mathrm{Span}(X_1) \cup \cdots \cup \mathrm{Span}(X_n)$

# The problem

$\Sigma$ finite alphabet

Identify $\mathbb{Q}^{d\times d}$ with $\mathbb{Q}^{d^2}$

**Def:** Lin CFL Closure problem

In: $L \subseteq \Sigma^*$ context free

$\mu : \Sigma^* \to \mathbb{Q}^{d\times d}$ monoid morphism

Out: $\overline{\mu(L)}^{\ell}$

**Prop:** Lin closure $\leqslant$ closure

**Proof:** $\overline{X}^{z} = X_1 \cup \cdots \cup X_n \Rightarrow \overline{X}^{\ell} = \mathrm{Span}(X_1) \cup \cdots \cup \mathrm{Span}(X_n)$

# But why ?

$(u, \mu, \nu)$ a WA

Computing the linear hull $\overline{u\mu(\Sigma^*)}^{\ell}$

- → deciding sequentiality & unambiguity [Bell & Smertnig, 23]
- → minimizing CRA [Benalioua, Lhote, Reynier, 24]

# But why?

$(u, \mu, v)$ a WA

Computing the linear hull $\overline{u\mu(\Sigma^*)}^{\ell}$

→ deciding sequentiality & unambiguity

[Bell & Smertnig, 23]

→ minimizing CRA

[Benalioua, Lhote, Reynier, 24]

Idea: $\overline{u\,\mu(\Sigma^*)}^{\ell} = u\,\overline{\mu(\Sigma^*)}^{\ell}$

Thm: [Bell & Smertnig, 23]

Given $X \subseteq \mathbb{Q}^{d \times d}$ a closed set

$\overline{\langle X \rangle}^{\ell}$ is computable

# But why?

$(u, \mu, v)$ a WA

Computing the linear hull $\overline{u\mu(\Sigma^*)}^{\ell}$

- deciding sequentiality & unambiguity
  [Bell & Smertnig, 23]
- minimizing CRA
  [Benalioua, Lhote, Reynier, 24]

Idea: $\overline{u\,\mu(\Sigma^*)}^{\ell} = u\,\overline{\mu(\Sigma^*)}^{\ell}$

Thm: [Bell & Smertnig, 23]

Given $X \subseteq \mathbb{Q}^{d \times d}$ a closed set

$\overline{\langle X \rangle}^{\ell}$ is computable

Cor: Lin REG closure is computable

Proof: by induction

# But why?

$(u, \mu, v)$ a WA

Computing the linear hull $\overline{u\mu(\Sigma^*)}^{\ell}$

→ deciding sequentiality & unambiguity
[Bell & Smertnig, 23]

→ minimizing CRA
[Benalioua, Lhote, Reynier, 24]

Idea: $\overline{u\,\mu(\Sigma^*)}^{\ell} = u\,\overline{\mu(\Sigma^*)}^{\ell}$

Thm: [Bell & Smertnig, 23]

Given $X \subseteq \mathbb{Q}^{d\times d}$ a closed set

$\overline{\langle X \rangle}^{\ell}$ is computable

Cor: Lin REG closure
is computable

Proof: by induction

- Lin CFL closure is a natural generalization
- Linked with the LH of WTA
- CFL closure considered in [Ait El Manssour et al. 26]

# What is known

Lin REG closure

computable

[Bell & Smertnig, 23]

# What is known

Lin REG closure computable [Bell & Smertnig, 23]

$\preceq$

REG closure computable [Hrushovski et al., 18]

# What is known

Lin REG closure
computable
[Bell & Smertnig, 23]

$\Leftarrow$

REG closure
computable
[Hrushovski et al., 18]

1VASS closure
computable

$\Leftarrow$

CFL closure
???

$\Leftarrow$

Indexed closure
uncomputable

[Ait El Manssour et al. 26]

# What is known

Lin REG closure computable [Bell & Smertnig, 23] $\preccurlyeq$ REG closure computable [Hrushovski et al., 18]

(Lin) 1VASS closure computable $\preccurlyeq$ (Lin) CFL closure ??? $\preccurlyeq$ (Lin) Indexed closure uncomputable

[Ait El Manssour et al. 26]

# What is known (from [Ait El Manssour et al. 26])

Thm: 1VASS closure
is computable

Idea: Approximation by
regular languages

→ pumping arguments
based on a variant of
Simon's factorization
theorem

# What is known (from [Ait El Manssour et al. 26])

**Thm:** 1VASS closure is computable

**Idea:** Approximation by regular languages

→ pumping arguments based on a variant of Simon's factorization theorem

**Thm:** [Dufourd et al., 98]

$\underbrace{\text{Boundedness}}_{\text{set of config. finite ?}}$ of reset VASS is undecidable

**Thm:** Indexed closure is uncomputable

**Idea:** Encode runs as products of matrices generated by an indexed grammar

## What is known

**Thm:** if $L \subseteq GL_d(\mathbb{Q})$ then

Lin CFL closure is

computable

## ⚠ Ongoing work

## What is known

## ⚠ Ongoing work

**Thm:** if $L \subseteq GL_d(\mathbb{Q})$ then Lin CFL closure is computable

**Idea:** Lin CFL closure

$$\Updownarrow$$

computing LH of WTA

& LH of WTA is computable given a bound on its length

# What is known

**Thm:** if $L \subseteq GL_d(\mathbb{Q})$ then Lin CFL closure is computable

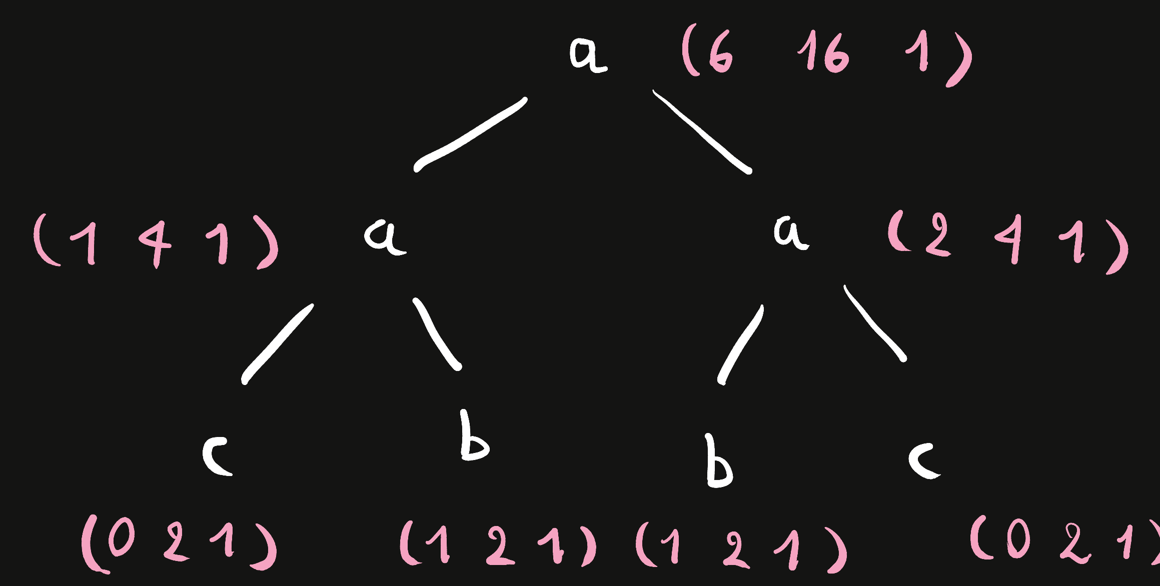
**Idea:** Lin CFL closure

$\Uparrow$

computing LH of WTA

& LH of WTA is computable given a bound on its length

# ⚠ Ongoing work



# What is known

**Thm:** if $L \subseteq GL_d(\mathbb{Q})$ then
Lin CFL closure is
computable

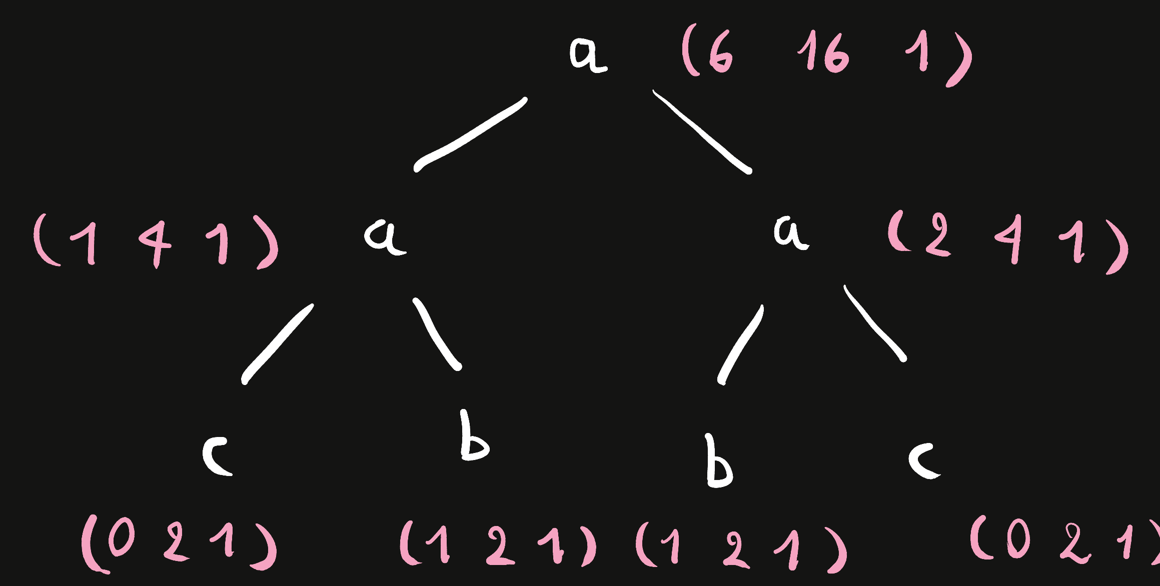
**Idea:** Lin CFL closure

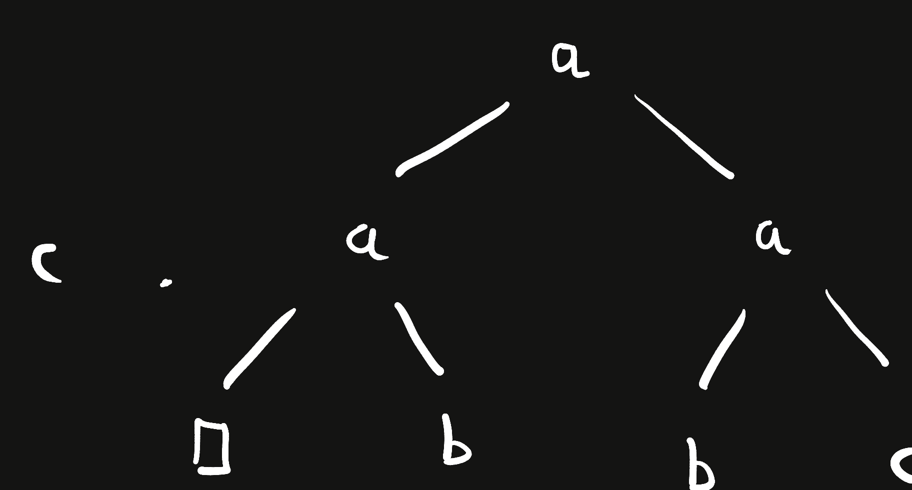
$\Uparrow$

computing LH
of WTA

& LH of WTA is computable
given a bound on its length

# ⚠ Ongoing work




$$(0\ 2\ 1)\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 6 & 0 & 1 \end{pmatrix} = (6\ \ 16\ \ 1)$$

## What is known

## ⚠ Ongoing work

**Def:** stable closure

**In:** $\hat{F}$ family of linear maps $\mathbb{Q}^{d\times d} \to \mathbb{Q}^{d\times d}$

**Out:** smallest submonoid of $\mathbb{Q}^{d\times d}$ closed & stable by $\hat{F}$

**Thm:** computing LH of WTA (binary trees) $\iff$ stable closure

## What is known

## ⚠ Ongoing work

**Def:** stable closure

**In:** $\hat{F}$ family of linear maps $\mathbb{Q}^{d\times d} \to \mathbb{Q}^{d\times d}$

**Out:** smallest submonoid of $\mathbb{Q}^{d\times d}$ closed & stable by $\hat{F}$

**Thm:** computing LH of WTA (binary trees) $\iff$ stable closure

**Ex:** $\tilde{F}$ contains only $M \mapsto AMB$

$\iff$ Dyck closure

# Questions

- (Lin) CFL closure $\lessapprox$ (Lin) Dyck closure ?

# Questions

- (Lin) CFL closure $\leq$ (Lin) Dyck closure ?
- Bound the length of $\overline{\langle C_I \rangle}^{\ell}$ ?

# Questions

- (Lin) CFL closure $\leq$ (Lin) Dyck closure ?
- Bound the length of $\overline{\langle C_I \rangle}^{\ell}$ ?
- Example of infinite chain for stable closure ?

# Questions

- (Lin) CFL closure $\leqslant$ (Lin) Dyck closure ?
- Bound the length of $\overline{\langle C_I \rangle}^{\ell}$ ?
- Example of infinite chain for stable closure ?
- Approximation by regular languages ?

# Questions

- (Lin) CFL closure $\leq$ (Lin) Dyck closure ?
- Bound the length of $\overline{\langle C_I \rangle}^{\ell}$ ?
- Example of infinite chain for stable closure ?
- Approximation by regular languages ?
- Is it uncomputable ?

# Questions

Thank you
for your attention

- (Lin) CFL closure $\leqslant$ (Lin) Dyck closure ?
- Bound the length of $\overline{\langle C_I \rangle}^{\ell}$ ?
- Example of infinite chain for stable closure ?
- Approximation by regular languages ?
- Is it uncomputable ?